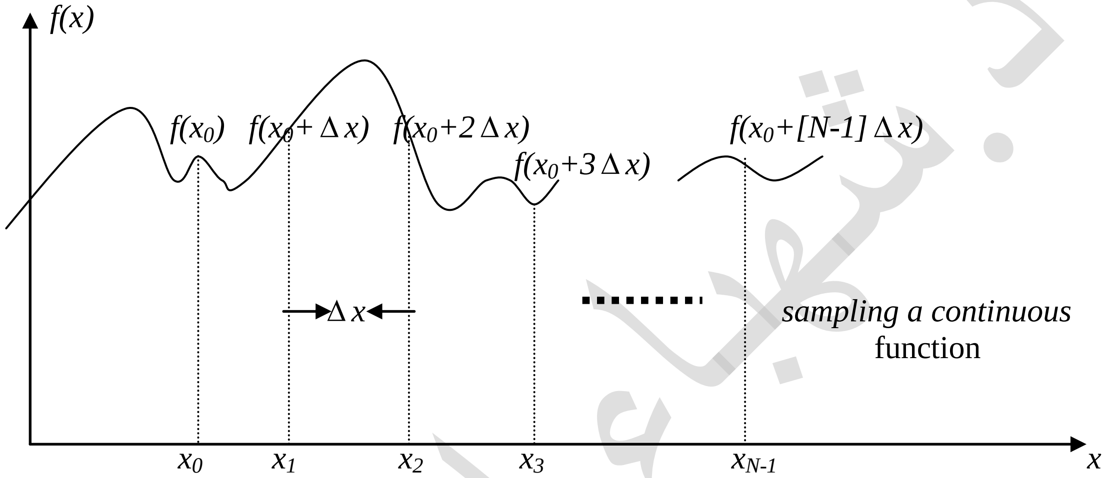
The Discrete Fourier Transform

Suppose that a continuous function $f(x)$ is discretized into a sequence $\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots\ldots\ldots\ldots, f(x_0+[N-1]\Delta x)\}$ by taking N samples Δx units a part, as shown in the following figure. It will be convenient in subsequent developments to use *x* as either a discrete or continuous variable, we may do this by defining: $f(x) = f(x_0 + x\Delta x)$



Where *x* now assumes the discrete values *0,1,2 , ... , N-1* . In other words the sequence $\{f(0), f(1), f(2), \ldots\ldots\ldots\ldots, f(N-1)\}$ will be used to denote any N uniformly spaced samples from a corresponding continuous function. The discrete fourier transform pair that applies to sampled functions is given by:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi\, ux/N] \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

for *u=0,1,2, . . . , N-1* and

$$f(x) = \sum_{u=0}^{N-1} F(u)\exp[j2\pi\, ux/N] \qquad \ldots\ldots\ldots\ldots\ldots..(2)$$

for *x=0,1,2 , . . . , N-1*

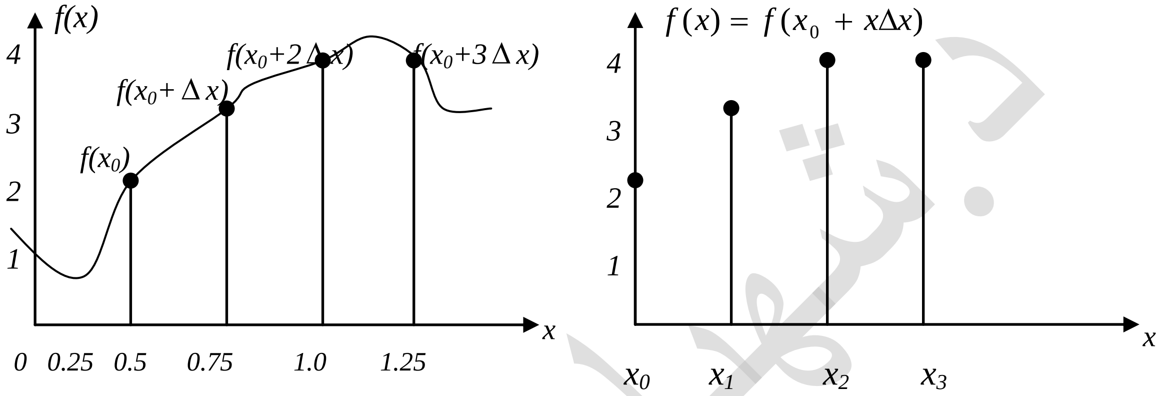
The values *u=0,1,2, . . . ,N-1* in the discrete fourier transform given in Eq. (1) correspond to samples of the continuous transform at values $0, \Delta u, 2\Delta u, \ldots\ldots, (N-1)\Delta u$

In other words we are letting *F(u)* represent $F(u\Delta u)$. This notation is similar to that used for the discrete *f(x)*, with the exception that the samples of

F(u) start at the origin of the frequency axis. It can be shown that Δu and Δx are related by the expression: $\Delta u = \dfrac{1}{N\Delta x}$

Example :

As an illustration of Eqs (1) and (2) consider the function shown in figure (a). If this function is sampled at the argument values x_0=0.5 , x_1=0.75 , x_2=1.0 , x_3=1.25 and if the argument is redefined as discussed above, we obtain the discrete function shown in fig. (b).



(a) x is continuous

(b) x is discrete

Application of Eq. (1) to the resulting four samples yields the following sequence of steps:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi\, ux/N]$$

$$F(0) = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi\, x*0/4]$$

$$F(0) = \frac{1}{4}\sum_{x=0}^{3} f(x)\, e^0$$

$$= \frac{1}{4}\,[\, f(0) + f(1) + f(2) + f(3)\,]$$

$$= \frac{1}{4}\,[\, 2 + 3 + 4 + 4\,]$$

$$= 3.25$$

$$F(1) = \frac{1}{4}\sum_{x=0}^{3} f(x)\ \exp[-j2\pi\, x*1/4]$$

$$= \frac{1}{4}[\,2\ e^0 + 3\ e^{-j\pi/2} + 4\ e^{-j\pi} + 4\ e^{-j3\pi/2}]$$

$$= \frac{1}{4}[2 + 3\,(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}) + 4\,(\cos\pi - j\sin\pi) + 4\,(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2})]$$

$$= \frac{1}{4}\ [2 + 3\,(0 - j) + 4\,(-1 - 0) + 4\,(0 + j)]$$

$$= \frac{1}{4}\ [\,2 - 3j - 4 + 4j\,]$$

$$= \frac{1}{4}[\ -2 + j\ ]$$

$$F(2) = \frac{1}{4}\sum_{x=0}^{3} f(x)\ \exp[-j4\pi\ x/4]$$

$$= \frac{1}{4}[\ 2\ e^{0}\ + 3\ e^{-j\pi}\ + 4\ e^{-j2\pi}\ + 4\ e^{-j3\pi}\ ]$$

$$= -\frac{1}{4}[1 + 0\,j\ ]$$

and

$$F(3) = \frac{1}{4}\sum_{x=0}^{3} f(x)\ \exp[-j6\pi\ x/4]$$

$$= \frac{1}{4}[\ 2\ e^{0}\ + 3\ e^{-j3\pi/2}\ + 4\ e^{-j3\pi}\ + 4\ e^{-j9\pi/2}]$$

$$= -\frac{1}{4}[2 +\ j\ ]$$

The Fourier spectrum is obtained from the magnitude of each of the transform terms, that is :

$|F(0)| = 3.25$

$|F(1)| = [(2/4)^2 + (1/4)^2]^{1/2} = \sqrt{5}/4$

$|F(2)| = [(1/4)^2 + (0/4)^2]^{1/2} = 1/4$

$|F(3)| = [(2/4)^2 + (1/4)^2]^{1/2} = \sqrt{5}/4$

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In the two-variable case the discrete Fourier transform pair is given by the equations:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(\,ux/M + vy/N)]$$

for u=0,1,2, . . . , M-1 , v=0,1,2, . . . ,N-1

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[\,j2\pi(\,ux/M + vy/N)]$$

for x=0,1,2 , . . . , M-1 and y=0,1,2 , . . . , N-1

sampling of a continuous function is now in a two-dimensional grid with divisions of width Δx and Δy in the x- and y- axis respectively. As in the one-dimensional case, the discrete function $f(x,y)$ represents samples of the function $f(x_0 + x\Delta x, y_0 + y\Delta y)$ for *x=0,1,2,…,M-1* and y*=0,1,2,…,N-1* similar comments hold for *F(u,v)*. The sampling increments in the spatial and frequency domains are related by:

$$\Delta u = \frac{1}{M\Delta x} \quad \text{and} \quad \Delta v = \frac{1}{N\Delta y}$$

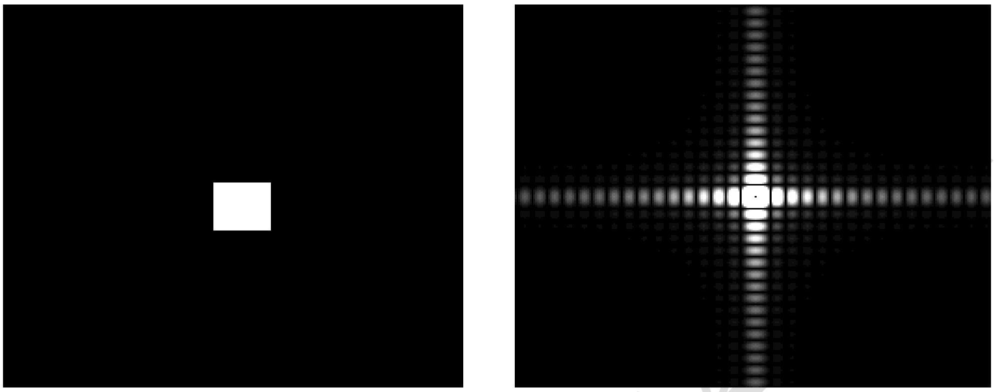
when images are sampled in a square array we have that *M=N* and

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\exp[-j2\pi(ux+vy)/N]$$

for u,v=0,1,2, . . . ,N-1

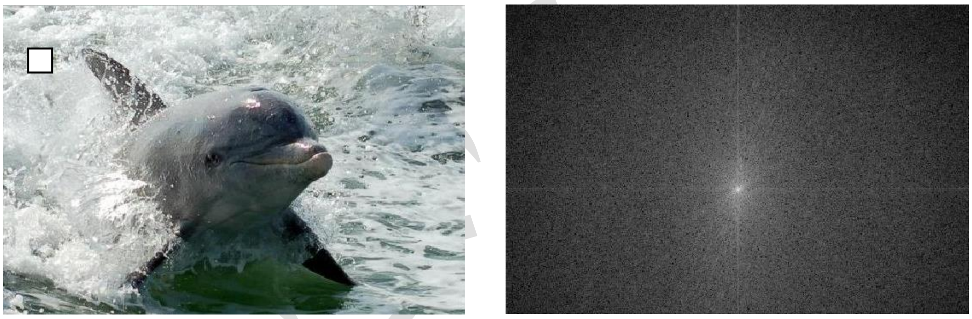
$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F(u,v)\exp[j2\pi(ux+vy)/N]$$

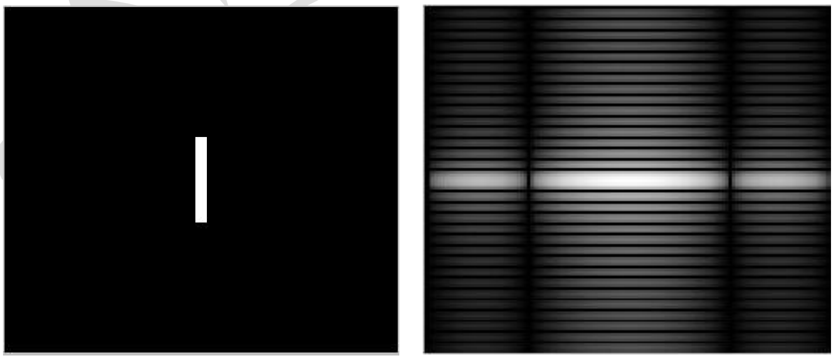
for x,y=0,1,2 , . . . , N-1

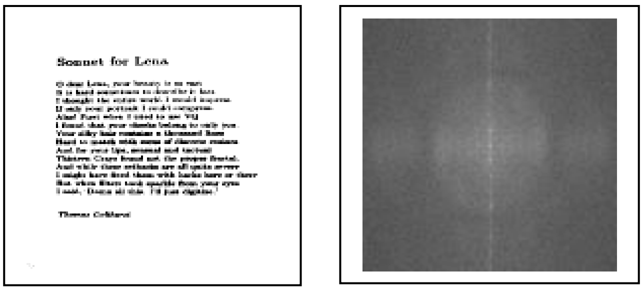


(a) *(b)*

Figure: (a) square image , (b) DFT magnitude of the square image







***Example:* Find the DFT of the following function**

$$f(x,y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}_{4*4}$$

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux+vy)/N]$$

$$\begin{aligned} F(0,0) = \frac{1}{4}[(&f(0,0)\ e^{-j2\pi(u*0+v*0)/N} + f(0,1)\ e^{-j2\pi(u*0+v*1)/N} + \\ &f(0,2)\ e^{-j2\pi(u*0+v*2)/N} + f(0,3)\ e^{-j2\pi(u*0+v*3)/N}) \\ &+ (f(1,0)\ e^{-j2\pi(u*1+v*0)/N} + f(1,1)\ e^{-j2\pi(u*1+v*1)/N} + \\ &f(1,2)\ e^{-j2\pi(u*1+v*2)/N} + f(1,3)\ e^{-j2\pi(u*1+v*3)/N}) \\ &+ (f(2,0)\ e^{-j2\pi(u*2+v*0)/N} + f(2,1)\ e^{-j2\pi(u*2+v*1)/N} + \\ &f(2,2)\ e^{-j2\pi(u*2+v*2)/N} + f(2,3)\ e^{-j2\pi(u*2+v*3)/N}) \\ &+ (f(3,0)\ e^{-j2\pi(u*3+v*0)/N} + f(3,1)\ e^{-j2\pi(u*3+v*1)/N} + \\ &f(3,2)\ e^{-j2\pi(u*3+v*2)/N} + f(3,3)\ e^{-j2\pi(u*3+v*3)/N})] \end{aligned}$$

$$\begin{aligned} = \frac{1}{4}[\ &f(0,0) + f(0,1) + f(0,2) + f(0,3) \\ &+ f(1,0) + f(1,1) + f(1,2) + f(1,3) \\ &+ f(2,0) + f(2,1) + f(2,2) + f(2,3) \\ &+ f(3,0) + f(3,1) + f(3,2) + f(3,3)] \end{aligned}$$

$$= \frac{1}{4}\ [\ 0 + 0 + 1 + 0 + 2 + 0 + 0 + 0 + 0 + 3 + 0 + 0 + 0 + 1 + 0 + 2]$$

$$= \frac{1}{4}\ [\ 9]$$

$$= 2.25$$