properties of the two-dimensional Fourier transform

Convolution

The *DFT* of convolution of two functions is equal to the product of the DFT of these two functions, i.e.

$\Im\{ f_1(x,y) * f_2(x,y) \} = F_1(u,v) \cdot F_2(u,v)$

The convolution of two functions *f(x)* and *g(x)* denoted by *f(x)*g(x)* is defined by the integral:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$

Where α is a dummy variable of integration. The convolution of two functions *f(x)* and *g(x)* before carrying out the integration it is necessary to form the function $g(x-\alpha)$. It is noted that this operation is simply one of folding $g(\alpha)$ about the origin to give $g(-\alpha)$ and then displacing this function by *x*. Then for any given value of *x,* we multiply $f(\alpha)$ by the corresponding $g(x-\alpha)$ and integrate the product from $-\infty$ to ∞.

Suppose that instead of being continuous f(x) and g(x) are discretized into sampled arrays of size A and B, respectively:

$\{f(0), f(1), f(2), \ldots\ldots\ldots\ldots, f(A-1)\}$ *, and*

$\{g(0), g(1), g(2), \ldots\ldots\ldots\ldots, g(B-1)\}$.

The discrete Fourier transform and its inverse are periodic functions. In order to formulate a discrete convolution theorem that is consistent with this periodicity property , we may assume that the discrete functions *f(x)* and *g(x)* are periodic with some period M. the resulting convolution will then be periodic with the same period. The value of *M* is selected as: *M=A+B-1.*

Since the assumed period must be greater than either *A* or *B*. the length of the sampled sequences must be increased so that both are of length *M*. this can be done by appending zeros to the given samples to form the following extended sequences:

$$f_e(x) = \begin{cases} f(x) & 0 \le x \le A-1 \\ 0 & A \le x \le M-1 \end{cases}$$

and

$$g_e(x) = \begin{cases} g(x) & 0 \le x \le B-1 \\ 0 & B \le x \le M-1 \end{cases}$$

Based on this we define the discrete convolution of $f_e(x)$ *and* $g_e(x)$ by the expression:

$$f_e(x) * g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

for x=0,1,2, …. , M-1. The convolution function is a discrete, periodic array of length *M* , with the values *x=0,1,2 , … , M-1* describing a full period of $f_e(x)*g_e(x)$.

Example: Convolved the following functions

f = { 3 , 2.5 , 1 , 0.5 }

g = { 2 , 5 , 7 , 9 }

f contains 4 elements

g contains 4 elements

$M = A + B - 1 = 4 + 4 - 1 = 7$ *the interval 0 . . . 6*

$$h_e(x) = f_e(x) * g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

$$h_e(0) = f_e(0).g_e(0) + f_e(1).g_e(-1) + f_e(2).g_e(-2) + f_e(3).g_e(-3) + f_e(4).g_e(-4) + f_e(5).g_e(-5) + f_e(6).g_e(-6)$$
$$= 3 * 2 = \mathbf{6}$$

$$h_e(1) = f_e(0).g_e(1) + f_e(1).g_e(0) + f_e(2).g_e(-1) + f_e(3).g_e(-2) + f_e(4).g_e(-3) + f_e(5).g_e(-4) + f_e(6).g_e(-5)$$
$$= 3 * 5 + 2.5 * 2 = 15 + 5 = \mathbf{20}$$

$$h_e(2) = f_e(0).g_e(2) + f_e(1).g_e(1) + f_e(2).g_e(0) + f_e(3).g_e(-1) + f_e(4).g_e(-2) + f_e(5).g_e(-3) + f_e(6).g_e(-4)$$
$$= 3 * 7 + 2.5 * 5 + 1 * 2 = 21 + 12.5 + 2 = \mathbf{35.5}$$

$$h_e(3) = f_e(0).g_e(3) + f_e(1).g_e(2) + f_e(2).g_e(1) + f_e(3).g_e(0) + f_e(4).g_e(-1) + f_e(5).g_e(-2) + f_e(6).g_e(-3)$$
$$= 3 * 9 + 2.5 * 7 + 1 * 5 + 0.5 * 2 = 27 + 17.5 + 5 + 1 = \mathbf{50.5}$$

$$h_e(4) = f_e(0).g_e(4) + f_e(1).g_e(3) + f_e(2).g_e(2) + f_e(3).g_e(1) + f_e(4).g_e(0) + f_e(5).g_e(-1) + f_e(6).g_e(-2)$$
$$= 3 * 0 + 2.5 * 9 + 1 * 7 + 0.5 * 5 + 0 * 2 = \mathbf{32}$$

$$h_e(5) = f_e(0).g_e(5) + f_e(1).g_e(4) + f_e(2).g_e(3) + f_e(3).g_e(2) + f_e(4).g_e(1) + f_e(5).g_e(0) + f_e(6).g_e(-1)$$
$$= 3 * 0 + 2.5 * 0 + 1 * 9 + 0.5 * 7 = 9 + 3.5 = \mathbf{12.5}$$

$$h_e(6) = f_e(0).g_e(6) + f_e(1).g_e(5) + f_e(2).g_e(4) + f_e(3).g_e(3) + f_e(4).g_e(2) + f_e(5).g_e(1) + f_e(6).g_e(0)$$
$$= 0.5 * 9 = \mathbf{4.5}$$

- by matrix

*Matrix = M*M*

$$
\begin{pmatrix}
g_e(0) & g_e(-1) & g_e(-2) & \cdots & g_e(-M+1) \\
g_e(1) & g_e(0) & g_e(-1) & \cdots & g_e(-M+2) \\
g_e(2) & g_e(1) & g_e(0) & \cdots & g_e(-M+3) \\
\vdots & \vdots & \vdots & & \vdots \\
g_e(M-1) & g_e(M-2) & g_e(M-3) & \cdots & g_e(0)
\end{pmatrix}
$$

$$
\begin{pmatrix}
g_e(0) & g_e(M-1) & g_e(M-2) & \cdots & g_e(1) \\
g_e(1) & g_e(0) & g_e(M-1) & \cdots & g_e(2) \\
g_e(2) & g_e(1) & g_e(0) & \cdots & g_e(3) \\
\vdots & \vdots & \vdots & & \vdots \\
g_e(M-1) & g_e(M-2) & g_e(M-3) & \cdots & g_e(0)
\end{pmatrix}
$$