Integral of rational fun. Contain sin,cos **تكاملات نسبية حاوية على دوال مثلثية**

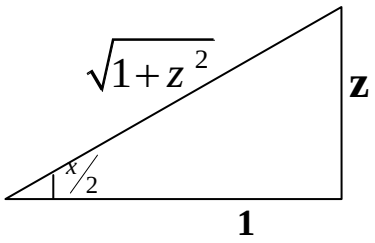
اذا ظهر sin , cos في دالة نسبية ولم نستطع تكاملها بالطرق الاعتيادية البسيطة السابقة نلجأ الى طرق التكامل بعدها نعوض عن قيم sin , cos بما يعادلها بالمتغير z ثم نجري عملية التكامل وبعد الانتهاء من عملية التكامل نعوض عن قيمة z بـ x .

$$z = \tan\frac{x}{2} \Rightarrow x = 2\tan^{-1} z$$

$$dx = \frac{2}{1+z^2}dz$$

$$\sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$



EXAM :

1 $\int \frac{dx}{1+\sin x}$

$$= \int \frac{\frac{2dz}{1+z^2}}{1+\frac{2z}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{1+z^2+2z}{1+z^2}} = 2\int \frac{dz}{(z+1)^2}$$

$$-2(z+1)^{-1} = \frac{-2}{z+1} = \frac{-2}{\tan\frac{x}{2}+1} + C$$

$$\sin x = \frac{2z}{1+z^2}$$

$$dx = \frac{2}{1+z^2}dz$$

2 $\int \frac{\cos x}{1+\sin x}dx$

$$= Ln\left|1+\sin x\right| + C$$

3 $\int \frac{dx}{1+\sin x-\cos x}$

$$= \int \frac{\frac{2dz}{1+z^2}}{1+\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{1+z^2+2z-1+z^2}{1+z^2}}$$

$$= \int \frac{dz}{z^2+z}$$

$$\sin x = \frac{2z}{1+z^2}$$

$$dx = \frac{2}{1+z^2}dz$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$\frac{1}{z^2+z}=\frac{1}{z(z+1)}=\frac{A}{z}+\frac{B}{z+1}=\frac{A(z+1)+Bz}{z(z+1)}$$

$$1=Az+A+Bz \Rightarrow A=1\ ,\ A+B=0\Rightarrow B=-1$$

$$\therefore \int\frac{dx}{1+\sin x-\cos x}=\int\frac{1}{z^2+z}dz=\int(\frac{1}{z}-\frac{1}{z+1})dz$$

$$=Ln|z|-Ln|z+1|=Ln\left|\tan\frac{x}{2}\right|-Ln\left|\tan\frac{x}{2}+1\right|+C$$

4 $\int\frac{dx}{2+\sin x}$

$$=\int\frac{\frac{2dz}{1+z^2}}{2+\frac{2z}{1+z^2}}=\int\frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{2+2z^2+2z}{\cancel{1+z^2}}}=\int\frac{dz}{z^2+z+1}$$

$$=\int\frac{dz}{z^2+z+\frac{1}{4}+\frac{3}{4}}=\int\frac{dz}{(z+\frac{1}{2})^2+\frac{3}{4}}$$

$$Let\ \ u=z+\frac{1}{2}\Rightarrow du=dz$$

$$=\int\frac{du}{u^2+\frac{3}{4}}\Rightarrow Let\ \ u=\frac{\sqrt{3}}{2}\tan\theta\Rightarrow du=\frac{\sqrt{3}}{2}\sec^2\theta d\theta$$

$$\tan\theta=\frac{u}{\sqrt{3}/2}\Rightarrow\theta=\tan^{-1}\frac{u}{\sqrt{3}/2}$$

$$=\int\frac{\frac{\sqrt{3}}{2}\sec^2\theta d\theta}{\frac{3}{4}\tan^2\theta+\frac{3}{4}}=\frac{\sqrt{3}}{2}\frac{4}{3}\int\frac{\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}}=\frac{2}{\sqrt{3}}\theta=\frac{2}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}/2}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{(z+\frac{1}{2})}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{(\tan\frac{x}{2}+\frac{1}{2})}{\sqrt{3}/2}+C$$

5 $\int \frac{dx}{3\sin x + 4\cos x}$

$$= \int \frac{\frac{2dz}{1+z^2}}{3\frac{2z}{1+z^2} + 4\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{6z+4-4z^2}{1+z^2}} = \int \frac{2dz}{-4z^2+6z+4} = -\int \frac{dz}{2z^2-3z-2}$$

$$\frac{1}{2z^2-3z-2} = \frac{1}{(2z+1)(z-2)} = \frac{A}{(2z+1)} + \frac{B}{(z-2)} \Rightarrow A = \frac{-2}{5} \ , \ B = \frac{1}{5}$$

$$\therefore \int \frac{dx}{3\sin x + 4\cos x} = \frac{2}{5}\int \frac{dz}{(2z+1)} - \frac{1}{5}\int \frac{dz}{(z-2)}$$

$$= \frac{1}{5}Ln|2z+1| - \frac{1}{5}Ln|z-2| = \frac{1}{5}Ln\left|2\tan\tfrac{x}{2}+1\right| - \frac{1}{5}Ln\left|\tan\tfrac{x}{2}-2\right| + C$$

6 $\int \frac{dx}{3\sec x - 1}$

$$= \int \frac{\frac{2dz}{1+z^2}}{3\frac{1+z^2}{1-z^2} - 1} = \int \frac{\frac{2dz}{1+z^2}}{\frac{3+3z^2-1+z^2}{1-z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{2+4z^2}{1-z^2}} = \int \frac{1}{1+z^2}\cdot\frac{1-z^2}{1+2z^2}dz$$

$$\frac{1-z^2}{(1+z^2)(1+2z^2)} = \frac{Az+B}{1+z^2} + \frac{Cz+D}{1+2z^2}$$

$A = 0$, $B = -2$, $C = 0$, $D = 3$

$$\int \frac{dx}{3\sec x - 1} = \int \frac{-2}{1+z^2}dz + \int \frac{3}{1+2z^2}dz = -2\tan^{-1} z + \frac{3}{\sqrt{2}}\tan^{-1}(\sqrt{2}z)$$

$$= -2\tan^{-1}(\tan x/2) + \frac{3}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x/2) + C$$

HOME WORK

$\int \frac{dx}{2-\sin x}$

$\int \frac{\sec x}{1+\sin x}dx$

$\int \frac{\cot x}{1+\sin x}dx$

$\int \frac{2}{\sin x + \tan x}dx$

$\int \frac{\sin x}{2-\sin x}dx$

$\int \frac{dx}{3+2\cos x}$

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