The Sin and Cosine functions can be written as :

$$sinx = \frac{e^{ix} - e^{-ix}}{2i} \quad , \quad cosx = \frac{e^{ix} + e^{-ix}}{2}$$

DEF.: The inverse sine fun. Denoted by $\sin^{-1} x$ is defined to be the inverse of restricted sine fun. $\sin x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

DEF.: The inverse cosine fun. Denoted by $\cos^{-1} x$ is defined to be the inverse of restricted cosine fun. $\cos x \quad 0 \le x \le \pi$

DEF.: The inverse tangent fun. Denoted by $\tan^{-1} x$ is defined to be the inverse of restricted tangent fun. $\tan x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

DEF.: The inverse secant fun. Denoted by $\sec^{-1} x$ is defined to be the inverse of restricted secant fun. $\sec x \quad 0 \le x \le \pi \quad , x \ne \frac{\pi}{2}$

NOTE : The -1 in the expression for the inverse means " inverse " it does not mean reciprocal , for example the reciprocal of sinx is $(sinx) = \frac{1}{sinx} = cscx$

$$\sin^{-1} x \ne \frac{1}{\sin x}$$

To get rid any trignometric function we will take the inverse function to the original function.
and To get rid any inverse trignometric function we will take the inverse function to the original function.
The inverse of inverse function is the function the same

EXAM :

(1) *if* $x = \sin^{-1}\frac{1}{2}$, *find value of* x

$$x = \sin^{-1}\frac{1}{2} \Rightarrow \sin x = \sin\sin^{-1}\frac{1}{2} \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$$

(2) *simplify* $\sin(\sin^{-1}\frac{1}{2})$

$$\because \sin^{-1}\frac{1}{2} = \frac{\pi}{6} \Rightarrow \sin(\sin^{-1}\frac{1}{2}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$$

(3) *show that* $\cos(\sin^{-1}\frac{1}{2}) = \frac{\sqrt{3}}{2}$

$$\because \cos x = \sqrt{1-\sin^2 x}$$

$$\therefore \cos(\sin^{-1} x) = \sqrt{1-\sin^2\sin^{-1}\frac{1}{2}} = \sqrt{1-\sin^2\frac{\pi}{6}} = \sqrt{1-(\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$$

Inverse Trigonometric Function:

1) $\sin^{-1}x = \frac{1}{i}\ln(ix+\sqrt{1-x^2})$

2) $\cos^{-1}x = \frac{1}{i}\ln(x+\sqrt{x^2-1})$

3) $\tan^{-1}x = \frac{1}{2i}\ln(\frac{i-x}{i+x})$

4) $\cot^{-1}x = \frac{1}{2i}\ln(\frac{x+i}{x-i})$

5) $\sec^{-1}x = \frac{1}{i}\ln(\frac{1+\sqrt{1-x^2}}{x})$

6) $\csc^{-1}x = \frac{1}{i}\ln(\frac{i+\sqrt{x^2-1}}{x})$

Proof 1:

$$let \;\; w = sin^{-1}(x)$$

$$\therefore \quad x = sin\, w$$

$x = \frac{e^{iw} - e^{-iw}}{2i}$ multiply two sides by $2ie^{iw}$, we get :

$$e^{2iw} - 2ixe^{iw} - 1 = 0$$

$$e^{iw} = \frac{2xi \mp \sqrt{-4x^2 + 4}}{2(1)}$$

$$e^{iw} = \frac{2xi \mp 2\sqrt{1 - x^2}}{2(1)}$$

$$e^{iw} = xi \mp \sqrt{1 - x^2}$$

$$iw = ln\left[xi + \sqrt{1 - x^2}\right]$$

$$w = \frac{1}{i} ln\left[xi + \sqrt{1 - x^2}\right] \rightarrow sin^{-1}x = \frac{1}{i} ln\left[xi + \sqrt{1 - x^2}\right]$$

Proof 3 :

$$let \;\; w = tan^{-1}(x)$$

$$\therefore \quad x = tan\, w$$

$$x = \frac{e^{iw} - e^{-iw}}{i(e^{iw} - e^{-iw})}$$

$ie^{iw}x - xe^{-iw} = e^{iw} - e^{-iw}$ multiply two sides by e^{iw} , we get :

$$ixe^{2iw} + ix = e^{2iw} - 1$$

$$e^{2iw}(1 - ix) = 1 + ix$$

$$e^{2iw}=\frac{1+ix}{1-ix}$$

$$e^{2iw}=\frac{i(x-i)}{-i(x+i)}$$

$$e^{2iw}=-\frac{(x-i)}{(x+i)}\rightarrow e^{2iw}=\frac{(i-x)}{(i+x)}$$

$$2iw=ln(\frac{(i-x)}{(i+x)})$$

$$w=\frac{1}{2i}ln(\frac{(i-x)}{(i+x)})$$

$$tan^{-1}(x)=\frac{1}{2i}ln(\frac{(i-x)}{(i+x)})$$

proof 2 ,4,5 and 6 is home work

Inverse Trigometric Function Properties

1) $\sin^{-1}(-x)=-\sin^{-1}(x)$

2) $\tan^{-1}(-x)=-\tan^{-1}(x)$

3) $\cos^{-1}(x)=\frac{\pi}{2}-\sin^{-1}(x)$

4) $\cot^{-1}(x)=\frac{\pi}{2}-\tan^{-1}(x)$

5) $\sec^{-1}(x)=\cos^{-1}(\frac{1}{x})$

6) $\sec^{-1}(-x)=\pi-\sec^{-1}(x)$

7) $\cot^{-1}(x)=\tan^{-1}(\frac{1}{x})$

8) $\csc^{-1}(x)=\sin^{-1}(\frac{1}{x})$

PROOF :

1) *let* $y=\sin^{-1}(-x)$

$\sin y=\sin\sin^{-1}(-x)=-x \Rightarrow -\sin y=x$

$\sin(-y)=x \Rightarrow -y=\sin^{-1}x \Rightarrow y=-\sin^{-1}x$

$\therefore \sin^{-1}(-x)=-\sin^{-1}x$

3) *let* $y=\frac{\pi}{2}-\sin^{-1}(x)$

$\sin^{-1}(x)=\frac{\pi}{2}-y \Rightarrow x=\sin(\frac{\pi}{2}-y)=\cos y \Rightarrow y=\cos^{-1}x$

$\therefore \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}(x)$

8) *let* $y=\sin^{-1}(\frac{1}{x})$

$\sin y=\frac{1}{x} \Rightarrow x=\frac{1}{\sin y}=\csc y \Rightarrow y=\csc^{-1}x$

$\therefore \csc^{-1}x=\sin^{-1}(\frac{1}{x})$

مشتقات الدوال المثلثية العكسية :

1. $\frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
2. $\frac{d}{dx}\cos^{-1}u=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
3. $\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$
4. $\frac{d}{dx}\cot^{-1}u=\frac{-1}{1+u^2}\frac{du}{dx}$
5. $\frac{d}{dx}\sec^{-1}u=\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$
6. $\frac{d}{dx}\csc^{-1}u=\frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Proof 1 :

$$\sin^{-1}x = \frac{1}{i} \ln\left[xi + \sqrt{1 - x^2}\right]$$

$$(\sin^{-1}x)' = \frac{1}{i} \; \frac{i + \dfrac{-2x}{2\sqrt{1-x^2}}}{xi + \sqrt{1-x^2}}$$

$$(\sin^{-1}x)' = \frac{1}{i} \; \frac{\dfrac{i\sqrt{1-x^2} - x}{\sqrt{1-x^2}}}{xi + \sqrt{1-x^2}}$$

$$(\sin^{-1}x)' = \frac{1}{i} \; \frac{i(\sqrt{1-x^2} + ix)}{\sqrt{1-x^2}} \; \frac{1}{xi + \sqrt{1-x^2}}$$

$$\therefore (\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$$

Proof 2,3,4,5 and 6 is Home Work

EXAM : Find $\frac{dy}{dx}$ **to :**

$$y = \sin^{-1} 2x \Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$$

$$y = \tan^{-1} 3x + e^{\tan^{-1} x} \Rightarrow \frac{dy}{dx} = \frac{3}{1+9x^2} + e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$y = \cos^{-1} \cos x \Rightarrow \frac{dy}{dx} = \frac{-(-\sin x)}{\sqrt{1-\cos^2 x}} = \frac{\sin x}{\sin x} = 1$$

$$OR \quad \cancel{\cos^{-1}}\, \cancel{\cos} x = x \Rightarrow \frac{dy}{dx} = 1$$

$$y = e^x \sec^{-1} x \Rightarrow \frac{dy}{dx} = e^x \cdot \frac{1}{x\sqrt{x^2-1}} + \sec^{-1} x \cdot e^x$$

تكاملات الدوال المثلثية العكسية :

$$\int \frac{du}{\sqrt{a^2-u^2}} = \begin{cases} \frac{1}{a}\sin^{-1}\frac{u}{a} \\ \\ \frac{-1}{a}\cos^{-1}\frac{u}{a} \end{cases} + C$$

$$\int \frac{du}{a^2+u^2} = \begin{cases} \frac{1}{a}\tan^{-1}\frac{u}{a} \\ \\ \frac{-1}{a}\cot^{-1}\frac{u}{a} \end{cases} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \begin{cases} \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| \\ \\ \frac{-1}{a}\csc^{-1}\left|\frac{u}{a}\right| \end{cases} + C$$

EXAM

$$\int \frac{dx}{1+16x^2} = \frac{1}{4}\int \frac{4dx}{1+(4x)^2} = \frac{1}{4}\tan^{-1}(4x)+C$$

$$\int \frac{dx}{9+4x^2} = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\frac{2x}{3}+C$$

$$\int \frac{e^x dx}{1+e^{2x}} = \int \frac{e^x dx}{1+(e^x)^2} = \tan^{-1}(e^x)+C$$

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{e^x dx}{e^x\sqrt{e^{2x}-1}} = \sec^{-1}\left|e^x\right|+C$$

$$\int \frac{dx}{\sqrt{x}\sqrt{4-x}} = 2\int \frac{dx}{2\sqrt{x}\sqrt{4-x}} = 2\cdot\frac{1}{2}\sin^{-1}\frac{\sqrt{x}}{2} = \sin^{-1}\frac{\sqrt{x}}{2}+C$$

$$\int \frac{x}{3+x^4}dx = \frac{1}{2}\int \frac{2x}{3+(x^2)^2}dx = \frac{1}{2}\frac{1}{\sqrt{3}}\tan^{-1}\frac{x^2}{\sqrt{3}} = \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x^2}{\sqrt{3}}+C$$

HOME WORK

1) Find $\frac{dy}{dx}$ to

$y = Ln(\cos^{-1} x)$, $y = \sqrt{\cot^{-1} x}$, $y = (\tan x)^{-1}$, $y = \cot^{-1}\sqrt{x}$

2) Find $\frac{dy}{dx}$ to

$x^3 + x\tan^{-1} y = e^y$, $\sin^{-1}(xy) = cos^{-1}(x - y)$

3) Find

$\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx$, $\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx$, $\int \frac{dx}{x\sqrt{1-Ln^2 x}}$

$\int \frac{\sin x}{\cos^2 x + 1}dx$, $\int \frac{1+\tan^2 x}{\sqrt{1-\tan^2 x}}dx$, $\int \frac{\sin x \cos x}{1+\cos^2(2x)}dx$

4) Find the area of the reigon bounded by the curve $x^2 y + 4y - 12 = 0$ **and x-axis .**

5) Find the volume generated by rotation about the x-axis , the reigon bounded by the curve $y = 5(x^2+1)^{-1/2}$, $y > 0$, $from \quad x = 0 , x = 4$

6) Find the length of the curve :

$y = \frac{1}{\sqrt{1-x^2}}$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

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