Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	Lecture sequences: 1	First lecture	Instructor Name: Dr. Saddam Atteyia
	The major contents: 1- Laplace Transformation (definition) 2- Linearity of the Laplace transformation 3- Some Functions and their Laplace Transform 4- Laplace Transform of Derivatives and Integrals		
	The detailed contents:		

Let $f(t)$ be a given function that is defined for all $t \geqq 0$. We multiply $f(t)$ by e^{-st} and integrate with respect to t from 0 to ∞. If the resulting integral exists, it is a function of s ($F(s)$):

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The function $F(s)$ of the variable s is called the Laplace transform of the original function $f(t)$, and will be denoted by $\mathcal{L}(f)$.

$$\therefore \quad \boxed{F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt} \qquad \text{—(1)}$$

The original function $f(t)$ in (1) is called the inverse transform or inverse of $F(s)$ and will be denoted by $\mathcal{L}^{-1}(F)$.

$$\therefore \quad \boxed{f(t) = \mathcal{L}^{-1}(F)}$$

Note:

Original functions are denoted by lowercase letters and their transforms by the same letters in capitals.

original function	transforms
$f(t)$	$F(s)$
$y(t)$	$Y(s)$

Ex1: let $f(t) = 1$ when $t \geq 0$. Find $F(s)$.

Solution: $\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \int_0^\infty e^{-st} dt$

$$= \frac{e^{-st}}{-s}\Big|_0^\infty = -\frac{1}{s}\left(\cancel{\frac{1}{e^{s\infty}}} - \frac{1}{1}\right)$$

$$\mathcal{L}(1) = \frac{1}{s} \quad , \; s > 0$$

Ex2: Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant

solution: $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$

$$= \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = -\frac{1}{(s-a)}\left(\cancel{\frac{1}{e^{(s-a)\infty}}} - \frac{1}{1}\right)$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a} \quad , \; s-a > 0$$

Theorem 1 (Linearity of the Laplace transformation)

For any functions $f(t)$ and $g(t)$ whose Laplace transform exists and any constants a and b,

$$\boxed{\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}}$$

Ex3: Let $f(t) = \cosh at = (e^{at} + e^{-at})/2$. Find $\mathcal{L}(f)$.

Solution: From theorem 1

$$\mathcal{L}\left\{\tfrac{1}{2}e^{at} + \tfrac{1}{2}e^{-at}\right\} = \tfrac{1}{2}\mathcal{L}(e^{at}) + \tfrac{1}{2}\mathcal{L}(e^{-at})$$

From Ex 2

$$\mathcal{L}(\cosh at) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) \quad , \quad s > a \ (\geqq 0)$$

$$= \frac{1}{2}\left(\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right)$$

$$= \frac{1}{2}\left(\frac{s+\cancel{a}+s-\cancel{a}}{s^2+\cancel{as}-\cancel{as}-a^2}\right)$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}s}{s^2-a^2}\right) = \frac{s}{s^2-a^2}$$

<u>Ex4</u> : Let $F(s) = \dfrac{1}{(s-a)(s-b)}$, $a \neq b$. Find $\mathcal{L}^{-1}(F)$

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<u>Solution</u> : By partial fractions reduction (Unrepeated factor).

$$\frac{1}{(s-a)(s-b)} = \frac{A}{(s-a)} + \frac{B}{(s-b)}$$

$$A = \lim_{s \to a} \frac{\cancel{(s-a)} \cdot 1}{\cancel{(s-a)}(s-b)}$$

$$\boxed{A = \frac{1}{a-b}}$$

$$B = \lim_{s \to b} \frac{\cancel{(s-b)} \cdot 1}{(s-a)\cancel{(s-b)}}$$

$$\boxed{B = \frac{1}{b-a}}$$

<u>OR</u>

نوحد المقام للجهة اليمنى من المعادلة

$$= \frac{As - Ab + Bs - Ba}{(s-a)(s-b)}$$

معامل S

$0S = As + Bs$

$$0 = A + B \Rightarrow \boxed{A = -B}$$

$$1 = -Ab - Ba$$

$$1 = -(-B)b - Ba$$

$$1 = Bb - Ba$$

$$1 = B(b-a) \Rightarrow \boxed{B = \frac{1}{b-a}}$$

$$\therefore A = -\frac{1}{b-a}$$

$$\boxed{A = \frac{1}{a-b}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{a-b}}{(s-a)}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{b-a}}{(s-b)}\right\}$$

$$= \frac{1}{a-b}\left[\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-b}\right\}\right]$$

From Ex2

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)$$

حفظ

Table 5.1 Some functions $f(t)$ and their Laplace transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n=1,2,\dots)$	$\frac{n!}{s^{n+1}}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\frac{s}{s^2-a^2}$
10	$\sinh at$	$\frac{a}{s^2-a^2}$

Notes:

1- Formulas 1, 2, 3 are special cases of formula 4

2- Formula 4 follows from formula 5

3- $\Gamma(n+1) = n!$, where n is nonnegative integer

$\Gamma(n+1)$ is the gamma function

5.2 Laplace Transforms of Derivatives and Integrals (page 249)

$$\boxed{\mathcal{L}(f') = s\mathcal{L}(f) - f(0)} \quad (s > \gamma) \quad —(1)$$

proof. by definition and by integration by parts.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

let $u = e^{-st}$, $dv = f'(t)$

$du = -s e^{-st}$, $v = f(t)$

$$\mathcal{L}\{f'(t)\} = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty f(t)\, e^{-st}\,dt$$

$$= \Big[\cancelto{0}{e^{-s(\infty)}} f(\infty) - \cancelto{1}{e^{-s(0)}} f(0)\Big] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

By applying (1) to the second-order derivative $f''(t)$ we obtain

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)} \quad —(2)$$

Similarly,

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - s f'(0) - f''(0) \quad —(3)$$

Theorem 2 [Laplace transform of the derivative of any order n]

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)} \quad -(4)$$

Ex1: Let $f(t) = t^2$. Find $\mathcal{L}(f)$

$f(t) = t^2 \Rightarrow f(0) = 0$

$f'(t) = 2t \Rightarrow f'(0) = 0$

$f''(t) = 2 \Rightarrow f''(0) = 2$

by applying (2)

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - s\cancel{f(0)}^{0} - \cancel{f'(0)}^{0}$$

$$\mathcal{L}\{2\} = s^2 \mathcal{L}\{f(t)\}$$

$$\frac{2}{s} = s^2 \mathcal{L}\{f(t)\} \Rightarrow \mathcal{L}\{f(t)\} = \frac{2}{s^3}$$

Ex5: A differential equation. Initial value problem.

Solve the initial value problem

$y'' + 4y' + 3y = 0 \quad , \quad y(0) = 3 \; , \; y'(0) = 1 .$

solution: $\mathcal{L}(y'') + 4\mathcal{L}(y') + 3\mathcal{L}(y) = 0$

Let $Y(s) = \mathcal{L}(y)$

$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 3$

$\mathcal{L}(y'') = s^2 Y(s) - s\,y(0) - y'(0)$

$= s^2 Y(s) - 3s - 1$

$$s^2 Y(s) - 3s - 1 + 4(sY(s) - 3) + 3Y(s) = 0$$

$$s^2 Y(s) - 3s - 1 + 4sY(s) - 12 + 3Y(s) = 0$$

$$s^2 Y(s) + 4s Y(s) + 3Y(s) = 3s + 1 + 12$$

$$(s^2 + 4s + 3) Y(s) = 3s + 13$$

$$(s+3)(s+1) Y(s) = 3s + 13$$

$$Y(s) = \frac{3s+13}{(s+3)(s+1)} = \frac{A}{(s+3)} + \frac{B}{(s+1)}$$

$$\frac{A(s+1) + B(s+3)}{(s+3)(s+1)} = \frac{As + A + Bs + 3B}{(s+3)(s+1)}$$

معامل s

$$3s = As + Bs$$

$$3 = A + B \Rightarrow A = 3 - B$$

$$13 = A + 3B \Rightarrow 13 = (3 - B) + 3B$$

$$13 = 3 - B + 3B$$

$$13 = 3 + 2B$$

$$2B = 10 \Rightarrow B = \frac{10}{2} = 5$$

$$A = 3 - 5 = -2$$

$$Y(s) = \frac{-2}{(s+3)} + \frac{5}{(s+1)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{(s+3)} + \frac{5}{(s+1)}\right\}$$

$$= -2\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\}$$

$$y(t) = -2e^{-3t} + 5e^{-t}$$

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Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	Lecture sequences: 2	Second lecture	Instructor Name: Dr. Saddam Atteyia
	The major contents: 1- Laplace Transform of the Integral of a Function 2- Shifting on the s-axis 3- Differentiation and Integration of Transforms 4- Convolution : Integral Equations		
	The detailed contents:		

Laplace Transform of the integral of a function

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\,\mathcal{L}\{f(t)\}} \quad (s>0,\ s>\gamma) \qquad \text{—(5)}$$

and

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau} \qquad \text{—(6)}$$

Ex 7: Let $\mathcal{L}(f) = \dfrac{1}{s(s^2+\omega^2)}$. Find $f(t)$

From table 5.1

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{\omega}\,\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega}\mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= \frac{1}{\omega}\sin\omega t$$

Apply Eq. (6)

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right\} = \frac{1}{\omega}\int_0^t \sin\omega\tau\,d\tau$$

$$= \frac{1}{\omega}\left[\frac{-\cos\omega\tau}{\omega}\right]_0^t = \frac{1}{\omega^2}\left[-\cos\omega t + \cancel{\cos 0}^{\,1}\right]$$

$$= \frac{1}{\omega^2}\left[1-\cos\omega t\right]$$

5.3 Shifting on the s-axis : Replacing s by s − a in F(s).

$$\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)} \quad \text{—(1)}$$

and

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)} \quad \text{—(1*)}$$

Ex 1:

$$\mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{e^{2t} \cos\omega t\} = \frac{(s-2)}{(s-2)^2+\omega^2}$$

$$\mathcal{L}\{e^{t} \sinh 2t\} = \frac{2}{(s-1)^2-4}$$

EX2 : Solve the initial value problem

$$y'' + 2y' + 5y = 0 \quad , \quad y(0) = 2 \; , \; y'(0) = -4$$

$$s^2 Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 5Y(s) = 0$$

$$Y(s)\left[s^2+2s+5\right] = 2s-\cancel{4}+\cancel{4}$$

$$Y(s) = \frac{2s}{s^2+2s+5} = \frac{2s+2-2}{s^2+2s+1+4} = \frac{2(s+1)-2}{(s+1)^2+4}$$

$$= 2\,\frac{(s+1)}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

take $\mathcal{L}^{-1}$ for the both sides

$$y(t) = 2e^{-t}\cos 2t - e^{-t}\sin 2t = e^{-t}(2\cos 2t - \sin 2t)$$

5.5 Differentiation and integration of transforms (p286)

Differentiation of Transforms

$$\frac{d}{ds}(F(s)) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\cdot(-t)\,f(t)\,dt = -\int_0^\infty e^{-st}(t\,f(t))\,dt$$

The derivative of

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

with respect to s can be obtained by differentiating under the integral sign with respect to s, thus

$$F'(s) = -\int_0^\infty e^{-st}\left[t\,f(t)\right]dt.$$

$$\therefore \boxed{\mathcal{L}\{t\,f(t)\} = -F'(s)} \quad -(1) \qquad F'(s) = \frac{d}{ds}F(s)$$

Ex1: Find $\mathcal{L}(t\sin\omega t)$

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

$$\therefore \mathcal{L}(t \sin \omega t) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = -\frac{-\omega * 2s}{(s^2+\omega^2)^2}$$

$$= \frac{2\omega s}{(s^2+\omega^2)^2}$$

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الى الجدول

Find $\mathcal{L}(t \cos \omega t)$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}(t \cos \omega t) = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = -\frac{(s^2+\omega^2)\cdot 1 - s(2s)}{(s^2+\omega^2)^2}$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = \frac{-1}{t}\mathcal{L}^{-1}\{F'(s)\}}$$

used to find $\mathcal{L}^{-1}$ $\{\ln \text{ \& } \tan^{-1}\}$

اي ان $F(s)$ هي $\ln$ أو $\tan^{-1}$

<u>EX2</u>: find $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\}$

$$1+\frac{\omega^2}{s^2} = \frac{s^2+\omega^2}{s^2}$$

$$\therefore F(s) = \ln\frac{s^2+\omega^2}{s^2} \Rightarrow -F'(s) = -\frac{d}{ds}\left[\ln\frac{s^2+\omega^2}{s^2}\right]$$

$$= -\left\{\frac{s^2}{s^2+\omega^2}\right\}\cdot\left\{\frac{s^2(2s)-(s^2+\omega^2)2s}{s^4}\right\}$$

$$= -\frac{\cancel{s^2}}{s^2+\omega^2}\cdot\frac{(\cancel{2s^3}-\cancel{2s^3}-2s\omega^2)}{\cancel{s^4}\, s^2} = \frac{2\cancel{s}\omega^2}{\cancel{s^2}\, s\,(s^2+\omega^2)}$$

$$F'(s) = \frac{-2\omega^2}{s(s^2+\omega^2)}$$

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\} = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{-2\omega^2}{s(s^2+\omega^2)}\right\}$$

$$-2\omega\,\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = -2\omega\sin\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{-2\omega^2}{s^2+\omega^2}\right)\right\} = -2\omega\int_0^t \sin\omega\tau\,d\tau$$

$$= -2\cancel{\omega}\left[\frac{-\cos\omega\tau}{\cancel{\omega}}\right]_0^t$$

$$= -2\left[-\cos\omega t+1\right]$$

$$= -2\left[1-\cos\omega t\right]$$

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\} = \frac{-1}{t} * -2\left[1-\cos\omega t\right]$$

$$= \frac{2}{t}\left[1-\cos\omega t\right]$$

5.6 Convolution. Integral Equations

Theorem 1 (Convolution theorem)

Let $f(t)$ and $g(t)$ satisfy the existence theorem. Then the product of their transforms $F(s)$ and $G(s)$ is the transform $H(s)$ of the convolution $h(t)$ of $f(t)$ and $g(t)$, written $\mathcal{L}(f*g)(t)$ and defined by

$$\boxed{h(t) = (f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau} \quad \text{—(1)}$$

The convolution $f*g$ has the properties

التبادل

$f*g = g*f$ (commutative law)

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$f*(g_1+g_2) = f*g_1 + f*g_2$ (distributive law)

الترتيبا

$(f*g)*v = f*(g*v)$ (associative law)

$f*0 = 0*f = 0$

But

$1*g \neq g$

ex: if $g(t) = t$, then

$$(1*g)(t) = \int_0^t 1\cdot(t-\tau)\,d\tau = \frac{t^2}{2}$$

we will used the convolution to solve the integral equation only.

EX4 : P275

Solve the integral equation $y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

solution :

1 st step. Equation in terms of convolution

$$y(t) = t + y(t) * \sin t$$

Take Laplace transformation for both sides

$$Y(s) = \frac{1}{s^2} + Y(s)\cdot\frac{1}{s^2+1}$$

$$Y(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s^2} \Rightarrow Y(s)\left[\frac{\cancel{s^2+1} - \cancel{1}}{s^2+1}\right] = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2}\,\frac{s^2+1}{s^2} = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

Taking the inverse transform

$$y(t) = t + \frac{1}{6}t^3$$

5.7 Partial Fractions P278

The solution of the subsidiary equation $Y(s)$ was

$$Y(s) = \frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials in s.

Assumption

$F(s)$ and $G(s)$ have real coefficients and no common factors. The degree of $F(s)$ is lower than that of $G(s)$.

we have four cases

(Case 1) Unrepeated factor $(s-a)$ ✓

(case 2) Repeated factor $(s-a)^m$ ✓

Case 1 : Unrepeated factor $(s-a)$

$$Y = \frac{F}{G} \qquad \text{a fraction } \frac{A}{(s-a)}$$

$$A = \lim_{s \to a} \frac{(s-a)\,F(s)}{G(s)} \qquad \text{or by } A = \frac{F(a)}{G'(a)}$$

يتم اشتقاق G(s) نسبة الى s ثم يتم تعويض قيمة a

EX 1: $$Y(s) = \frac{F(s)}{G(s)} = \frac{s+1}{s^3+s^2-6s}$$

$$= \frac{s+1}{s(\underbrace{s^2+s-6}_{(s-2)(s+3)})} = \frac{A_1}{s} + \frac{A_2}{(s-2)} + \frac{A_3}{(s+3)}$$

$$A_1 = \lim_{s \to 0} \frac{\cancel{s}(s+1)}{\cancel{s}(s-2)(s+3)} = \frac{1}{-2*3} = -\frac{1}{6}$$

$$A_2 = \lim_{s \to 2} \frac{\cancel{(s-2)}(s+1)}{s\cancel{(s-2)}(s+3)} = \frac{3}{2*5} = \frac{3}{10}$$

$$A_3 = \lim_{s \to -3} \frac{\cancel{(s+3)}(s+1)}{s(s-2)\cancel{(s+3)}} = \frac{-2}{-3*-5} = \frac{-2}{15}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{6}}{s} + \frac{\frac{3}{10}}{(s-2)} + \frac{\frac{-2}{15}}{(s+3)} \right\}$$

$$y(t) = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$$

Case 2 : Repeated factor $(s-a)^m$

$$Y(s) = \frac{F(s)}{G(s)}$$

$$\frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots\cdots + \frac{A_1}{(s-a)}$$

$$A_m = \lim_{s \to a} \frac{(s-a)^m F(s)}{G(s)}$$

and the other constants are given by

$$A_k = \frac{1}{(m-k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s-a)^m F(s)}{G(s)} \right], \quad k = 1, \dots, m-1$$

A, B
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P 286

5. $$\frac{s}{(s+1)^2} = \frac{A_2}{(s+1)^2} + \frac{A_1}{(s+1)}$$

$$A_2 = \lim_{s \to -1} \frac{\cancel{(s+1)^2} \cdot s}{\cancel{(s+1)^2}} = -1$$

$$A_1 = \frac{1}{(2-1)!} \lim_{s \to -1} \frac{d}{ds} \left[\frac{\cancel{(s+1)^2} \cdot s}{\cancel{(s+1)^2}} \right] = \frac{1}{1} * 1 = 1$$

13. $$\frac{2s^2 - 3s}{(s-2)(s-1)^2} = \frac{A}{(s-2)} + \frac{B_2}{(s-1)^2} + \frac{B_1}{(s-1)}$$

system of differential equations

A 35. $y_1' = -y_2$, $y_2' = y_1$, $y_1(0) = 1$, $y_2(0) = 0$

$$sY_1(s) - \overset{1}{\boxed{y_1(0)}} = -Y_2(s) \quad \text{—— ①}$$

$$sY_2(s) - \overset{0}{\boxed{y_2(0)}} = Y_1(s) \quad \text{—— ②}$$

from 2 $\quad sY_2(s) = Y_1(s)$

substitute in ①

$$s^2 Y_2(s) - 1 = -Y_2(s)$$

$$Y_2(s)[s^2+1] = 1$$

$$Y_2(s) = \frac{1}{s^2+1}$$

$$y_2(t) = \sin t$$

$$Y_1(s) = s\frac{1}{s^2+1}$$

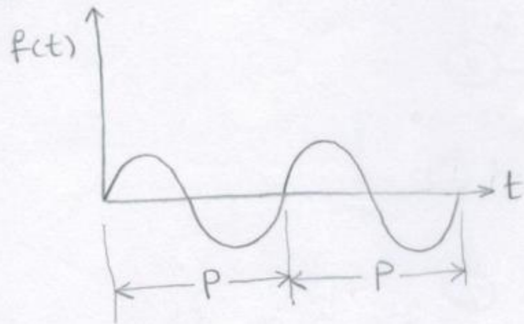
$$y_1(t) = \cos t$$

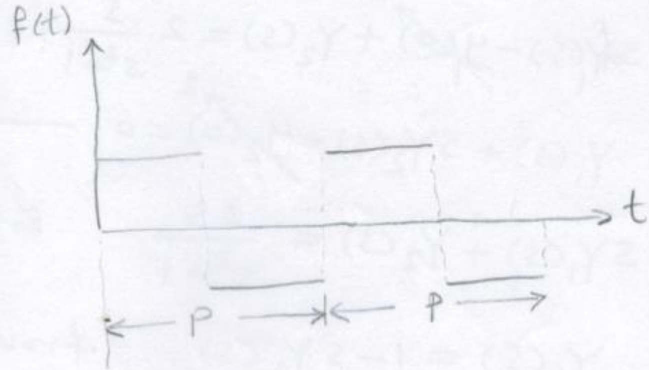
5.8 Periodic Functions Page 288

Let $f(t)$ be a function that is defined for all positive t and has the period $P\ (>0)$, that is

$$f(t+P) = f(t) \quad \text{for all } t > 0 .$$

Ex:



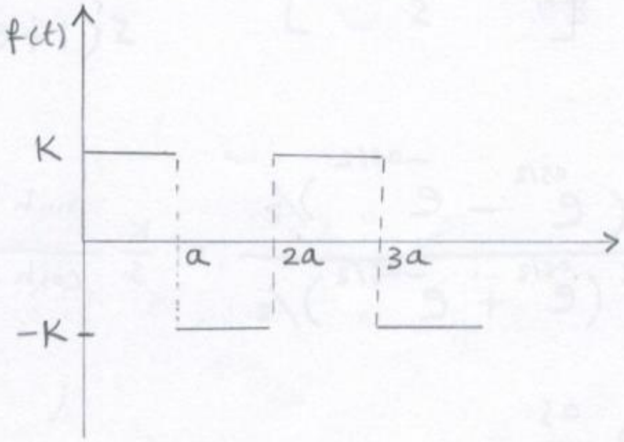


Theorem 1 (Transform of periodic functions)

The Laplace transform of a piecewise continuous periodic function $f(t)$ with period P is

$$\boxed{\mathcal{L}(f) = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\, dt} \quad (s>0) \quad —(1)$$

Ex1: Find the transform of the square wave



solution) $P = 2a$

$$\mathcal{L}(f) = \frac{1}{1-e^{-2as}} \left(\int_0^a K e^{-st} dt + \int_a^{2a} (-K) e^{-st} dt \right)$$

$$= \frac{1}{1-e^{-2as}} \left[K \frac{e^{-st}}{-s} \Big|_0^a - K \frac{e^{-st}}{-s} \Big|_a^{2a} \right]$$

$$= \frac{K}{1-e^{-2as}}\left[\frac{1}{-s}\left(e^{-as}-1\right)-\frac{1}{-s}\left(e^{-2as}-e^{-as}\right)\right]$$

$$= \frac{K}{1-e^{-2as}}\left[\frac{-e^{-as}+1}{s}-\frac{-e^{-2as}+e^{-as}}{s}\right]$$

$$= \frac{K}{1-e^{-2as}}\left[\frac{-e^{-as}+1+e^{-2as}-e^{-as}}{s}\right]$$

$$= \frac{K}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s}\right]$$

$$= \frac{K}{\cancel{(1-e^{-as})}(1+e^{-as})}\left[\frac{(1-e^{-as})^{2}}{s}\right] = \frac{K(1-e^{-as})}{s(1+e^{-as})}$$

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	**Lecture sequences: 3**	**Third lecture**	**Instructor Name: Dr. Saddam Atteyia**
	The major contents: **1- Fourier Series** **2- Functions of any Period P=2L** **3- Even and Odd Functions**		
	The detailed contents:		

10.2 Fourier series

ظهرت Fourier series نتيجة لتمثيل الدالة الدورية f(x) بسلسلة مثلثية Trigonometric series , ان سلسلة فورير للدالة f(x) هي سلسلة مثلثية معاملاتها توجد باستخدام الدالة f(x) وكما سيبين ادناه .

Euler Formulas of the Fourier coefficients

If $f(x)$ is a periodic function of period 2π, then it can be represented by a trigonometric series,

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)} \qquad (1)$$

The coefficients a_0, a_n, b_n of the series can be obtained by using Euler formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

الفترة هنا 2π

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, \dots$$

10.3 Functions of Any Period $p = 2L$

Periodic functions in applications rarely have period 2π but some other period $p = 2L$. If such a (يتطلب) function $f(x)$ has a Fourier series, we claim that it is of the form

لو اصبح للدالة $f(x)$ فترة $p = 2L$ بدلاً من 2π فان صيغة سلسلة فورير تصبح كما يلي (في التطبيقات الهندسية فان L قد يكون طول الوتر المهتز او طول القضيب المعدني عند انتقال الحرارة بالتوصيل)

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)}$$

with the Fourier coefficients of $f(x)$ given by the Euler formulas

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \dots$$

هذه الصيغة هي اكثر شمولاً من الحالة السابقة بحيث انه لو اصبح $L=\pi$ فسنحصل على السلسلة السابقة نفسها .

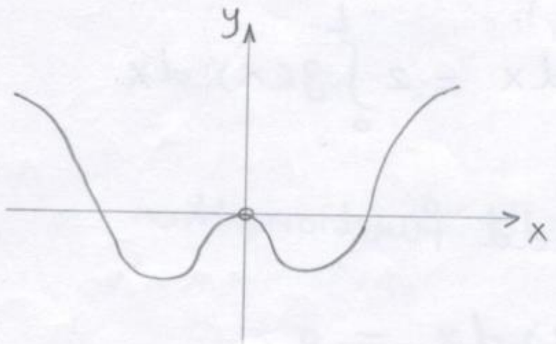
10.4 Even and Odd Functions

زوجية فردية

A function $y = g(x)$ is said to be even if

$$g(-x) = g(x) \qquad \text{for all } x$$

The graph of such a function is symmetric with respect to the y-axis



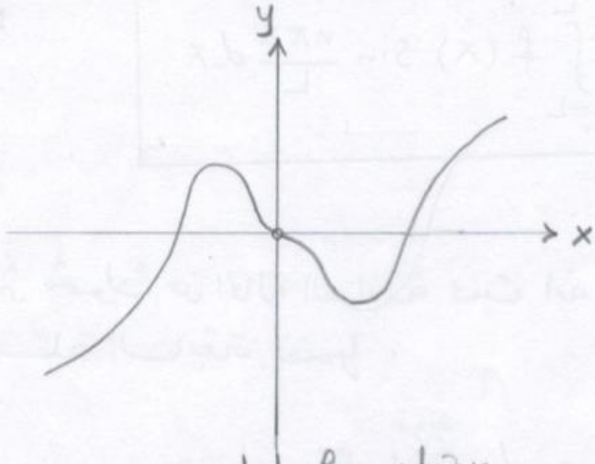
Even function

A function $h(x)$ is said to be odd if

$$h(-x) = -h(x) \qquad \text{for all } x$$

The graph of such a function is symmetric with respect to the origin

نقطة الاصل



odd function

Ex: The function $\cos nx$ is even
" " $\sin nx$ is odd

If $g(x)$ is an even function, then

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx \qquad (g \text{ even})$$

If $h(x)$ is an odd function, then

$$\int_{-L}^{L} h(x)\,dx = 0 \qquad (h \text{ odd})$$

The product of an even function g and an odd function h is odd

$$\underset{\text{odd}}{q} = \underset{\text{even}}{g}\ \underset{\text{odd}}{h}$$

because

$$q(-x) = g(-x)\,h(-x) = g(x)[-h(x)] = -q(x)$$

Hence if $f(x)$ is even, then $b_n = 0$. Similarly, if $f(x)$ is odd, then a_0 & $a_n = 0$. (هذا نتيجة الاسباب اعلاه)

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	**Lecture sequences: 4**	**Fourth lecture**	**Instructor Name: Dr. Saddam Atteyia**
	The major contents: **1- Fourier Series of Even and Odd Functions** **2- Half-Rang Expansions**		
	The detailed contents:		

Theorem 1 (Fourier series of even and odd functions)

The Fourier series of an even function $f(x)$ of Period $2L$ is a " Fourier cosine series"

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x}$$

with coefficients

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \ldots$$

The Fourier series of an odd function $f(x)$ of period $2L$ is a " Fourier sine series"

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x}$$

with coefficient

$$\boxed{b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx}$$

$\leftarrow$ odd * odd = even

2 * متناظر

Theorem 2 (Sum of functions)

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2

اي اذا كانت الدالة مجموع دالتين فأن معاملات فورير هي مجموع المعاملات للدالتين

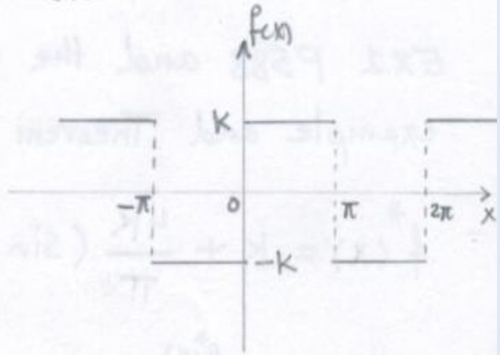
$$a_0 = a_{10} + a_{20}$$
$$a_n = a_{1n} + a_{2n}$$
$$b_n = b_{1n} + b_{2n}$$

(ضرب x)

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

Ex1. P 588

$$f(x) = \begin{cases} -K & \text{if} \quad -\pi < x < 0 \\ K & \text{if} \quad 0 < x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$



odd function

Solution: $P = 2\pi$

$$a_0 = a_n = 0$$
$$2L = 2\pi \Rightarrow L = \pi$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} K \sin \frac{n\cancel{\pi}x}{\cancel{\pi}} \, dx$$

$$= \frac{2K}{\pi} \left[\frac{-\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{2K}{n\pi} \left[-\overset{(-1)^n}{\cos n\pi} + \overset{1}{\cos 0} \right] = \frac{2K}{n\pi} \left[1 - (-1)^n \right]$$

$$= \frac{4K}{n\pi} \ (n \text{ is odd}) \ \ 1, 3, 5 \qquad = 0 \ (n \text{ is even}) \ \ 2, 4, 6$$

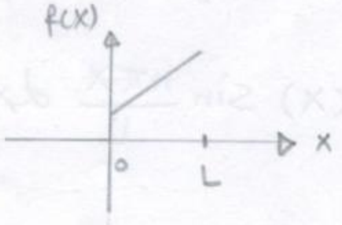
$$b_1 = \frac{4K}{\pi}, \; b_3 = \frac{4K}{3\pi}, \; b_5 = \frac{4K}{5\pi}, \; \cdots$$

$$\therefore f(x) = \sum_{n=1,3,5}^{\infty} b_n \sin nx$$

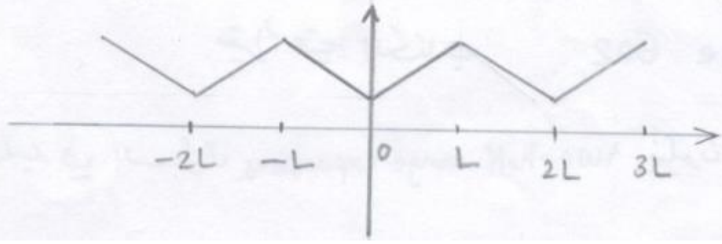
$$= \frac{4K}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

10.5 Half-Range Expansions

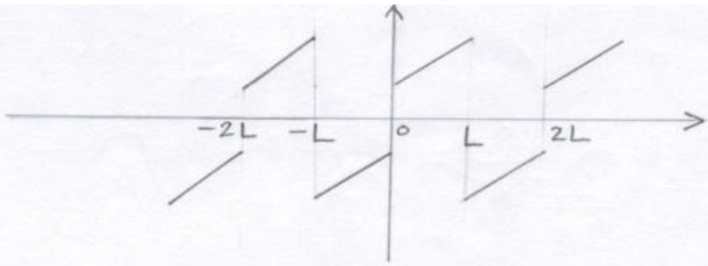
In various physical and engineering problems there is a practical need to use Fourier series with functions $f(x)$ that are given on some finite interval. Typical applications will arise in partial differential equations. Then $f(x)$ will be defined on some interval $0 \leqq x \leqq L$, and on this interval we want to represent $f(x)$ by a Fourier series. By choosing the period $2L$ we can get for $f(x)$ a Fourier cosine series, representing the even extension of $f(x)$ $(-L \leqq x \leqq L)$, or we can get for $f(x)$ a Fourier sine series, representing the odd extension of $f(x)$. These two series are called the two half-range expansions of $f(x)$.



The given function $f(x)$



$f(x)$ extended as an even function of $P=2L$



f(x) extended as an odd function of p=2L.

The form of these series is given in sec. 10.4. The cosine half-range expansion is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx \quad , n = 1, 2, \dots$$

The sine half-range expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \quad , \quad n = 1, 2, \dots$$

EX 1 Page 602 يقرأ في الكتاب

ملاحظة : اذا طلب في السؤال two half-range expansions فيكون الحل للحالتين معاً

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	**Lecture sequences: 5**	**Fifth lecture**	**Instructor Name: Dr. Saddam Atteyia**
	The major contents: **1- Special Functions (Gamma Function)** **2- Special Functions (Beta Function)**		
	The detailed contents:		

1- Gamma Function

Gamma function may be regarded as a generalization (تعميم) of the factorial function. It is defined by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n>0)$$

By integration by parts

$u = e^{-x}$ $\qquad dv = x^{n-1} dx$

$du = -e^{-x} dx$ $\qquad v = \frac{x^n}{n}$

$$= \cancel{\frac{e^{-x} x^n}{n}}\Big|_0^\infty + \int_0^\infty \frac{x^n}{n} e^{-x} dx$$

$$\Gamma(n) = \frac{1}{n}\Gamma(n+1) \qquad n<1 \text{ يستخدم عندما قيم}$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)} \qquad n>1 \text{ يستخدم عندما قيم}$$

$$\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$

$$\Gamma(2) = \Gamma(\overset{n}{1}+1) = \overset{n}{1}\,\Gamma(\overset{n}{1}) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2*1 = 2!$$

$$\Gamma(4) = \Gamma(3+1) = 3\,\Gamma(3) = 3 * 2 * 1 = 3!$$

$$\therefore \quad \boxed{\Gamma(n+1) = n!} \qquad (n = 0, 1, 2, \cdots)$$

n +ve integer لما

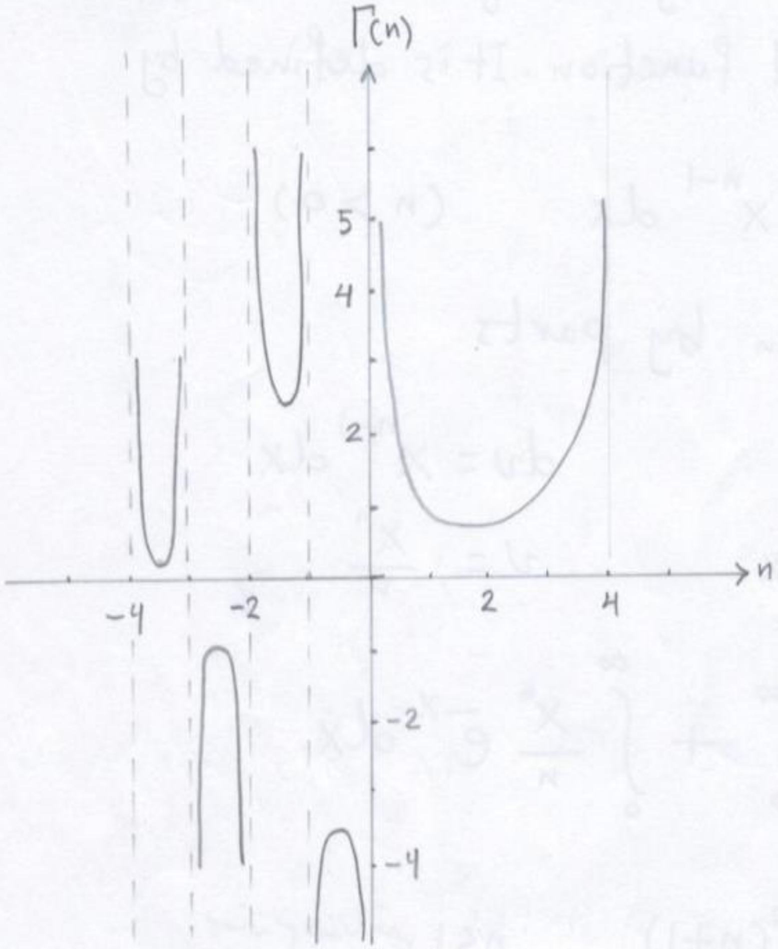


Fig. 545 Gamma Function
P A77

From Fig. 545 the function is not defined for $n = 0, -1, -2, \cdots$

Finally, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ تؤخذ بدون اشتقاق

Table A3 P A88 Gamma Function ($\alpha = n$)

هذا الجدول يعطي قيم Gamma function من n=1 الى n=2 وهي قيم دقيقة لانه يقسم n الى فترات صغيرة . اذا كانت هناك قيمة غير موجودة في الجدول فنجد القيمة بواسطة interpolation .

بعض القيم من الجدول

n	$\Gamma(n)$	n	$\Gamma(n)$
1.00	1.000 000	1.60	0.893515
1.10	0.951351	1.70	0.908639
1.20	0.918169	1.80	0.931384
1.30	0.897471	1.90	0.961766
1.40	0.887264	2.00	1.000 000
1.50	0.886227		

اكبر من 1

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

اقل من 1 ، اقل من 1

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\tfrac{3}{2}+1\right)}{-\tfrac{3}{2}} = \frac{\Gamma\left(-\tfrac{1}{2}\right)}{-\tfrac{3}{2}} = \frac{\Gamma\left(-\tfrac{1}{2}+1\right)}{-\tfrac{3}{2} * \left(-\tfrac{1}{2}\right)}$$

$$= \frac{4}{3}\Gamma\left(\tfrac{1}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

problems: Evaluate the value of the integral

1. $\int_0^\infty x^5 e^{-x}\,dx$

solution) compare with $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

$$n-1 = 5 \Rightarrow n = 6$$

$$\therefore \int_0^\infty x^5 e^{-x}\,dx = \Gamma(6) = 5! = 120$$

عدد صحيح وموجب

2. $\int_0^\infty 5^{-z^3}\, dz$

solution) $5^{-z^3} = e^{\ln 5^{-z^3}} = e^{-z^3 \ln 5}$

let $z^3 \ln 5 = u \Rightarrow e^{-z^3 \ln 5} = e^{-u}$

$$z^3 = \frac{u}{\ln 5} \quad \& \quad z = \frac{\sqrt[3]{u}}{\sqrt[3]{\ln 5}} = \frac{u^{1/3}}{\sqrt[3]{\ln 5}}$$

$$dz = \frac{1}{3} u^{-\frac{2}{3}} \cdot \frac{1}{\sqrt[3]{\ln 5}}\, du$$

$z = 0 \quad u = 0$

$z = \infty \quad u = \infty$

$$\int_0^\infty 5^{-z^3}\, dz = \int_0^\infty e^{-u} \cdot \frac{1}{3} \frac{u^{-\frac{2}{3}}}{\sqrt[3]{\ln 5}}\, du$$

$$= \frac{1}{3\sqrt[3]{\ln 5}} \int_0^\infty e^{-u} u^{-\frac{2}{3}}\, du$$

$n - 1 = -\frac{2}{3} \Rightarrow n = 1 - \frac{2}{3} = \frac{1}{3}$

$$= \frac{1}{3\sqrt[3]{\ln 5}} \Gamma\left(\frac{1}{3}\right) = \frac{1}{\cancel{3}\sqrt[3]{\ln 5}} \frac{\Gamma(\frac{1}{3}+1)}{\cancel{\frac{1}{3}}} = \frac{0.893024}{1.171902}$$

$$= 0.762029$$

Beta Function

It is denoted by $B(m,n)$ & it is defined as

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \qquad m,n > 0$$

Beta Function relations

1. $B(m,n) = B(n,m)$

2. $B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta$

3. $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

problems

1. Show that

$$\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \qquad m,n>0$$

solution) from beta function relations

$$2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$$

2. Show that

$$B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta$$

Solution : $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

let $x = \cos^2\theta \Rightarrow dx = 2\cos\theta(-\sin\theta)\,d\theta$

$= -2\cos\theta\sin\theta\,d\theta$

$\therefore \; 1-x = \sin^2\theta$

at $X = 0 \qquad \cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$X = 1 \qquad \cos^2\theta = 1 \Rightarrow \theta = 0$

$$B(m,n) = \int_{\pi/2}^{0} (\cos^2\theta)^{m-1} (\sin^2\theta)^{n-1} (-2\cos\theta \sin\theta \, d\theta)$$

$$= 2\int_0^{\pi/2} \cos^{2m-2}\theta \, \sin^{2n-2}\theta \, \cos\theta \sin\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta$$

3. Show that

$$\int_0^{\pi/2} \cos^K\theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{K+1}{2})}{\Gamma(\frac{K}{2}+1)} \quad , K > -1$$

From Beta function relations

$$B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$2m-1 = K$

$m = \frac{K+1}{2}$

&

$2n-1 = 0$

$n = \frac{1}{2}$

$$\therefore \int_0^{\pi/2} \cos^K\theta \, d\theta = \frac{\Gamma(\frac{K+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{K+1}{2}+\frac{1}{2})}$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{K+1}{2})}{\Gamma(\frac{K}{2}+1)}$$

Evaluate the following

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	**Lecture sequences: 6**	**Sixth lecture**	**Instructor Name: Dr. Saddam Atteyia**
	The major contents: 1- Partial Differential Equations (Basic Concepts) 2- One-Dimensional Wave Equation		
	The detailed contents:		

11.1 Basic Concepts

- An equation involving one or more partial derivatives of an (unknown) function of two or more independent variable is called a partial differential equation (PDE).

- The order of the highest derivative is called the order of the equation.

- PDE is linear if it is ① of the first degree in the dependent variable (the unknown function) and its partial derivatives. ② ايضاً يجب ان لايكون المتغير المعتمد مضروباً بمشتقته

- If each term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous; otherwise it is said to be nonhomogeneous.

Ex 1: Important linear PDE's of the second order

1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ 1D wave equation

2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ 1D heat equation

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 2D Laplace equation

4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ 2D poisson equation

5) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ 3D Laplace equation

where C is constant

t is time

x, y, z are Cartesian coordinates.

Eq. (4) with $f \neq 0$ is nonhomogeneous while the other equations are homogeneous.

- A solution of a PDE is a function that has all the partial derivatives appearing in the equation and satisfies the equation.

- There are many solutions to PDE. The unique solution of a PDE can be obtained by the use of additional information arising from the physical situation (boundary and initial conditions).

Boundary conditions: The values of the required solution of the problem on the boundary of some domine.

Initial conditions: In cases when time t is one of the variables, the values of the solution at $t=0$ will be prescribed.

The general form of 2nd order PDE of linear type is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu(x,y) = G(x,y)$$

where $A, \dots, G$ are functions of the independent variables (x, y) or constants

If $G(x,y)=0$, then the equation is called homogeneous.

The part $A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}=0$ is the principal part الاساسي

Types of PDE

from the principal part if

$B^2-4AC<0$ The equation is elliptic

$B^2-4AC=0$ The equation is Parabolic

$B^2-4AC>0$ The equation is hyperbolic

For the above equations, if we compare with the principal part

1- The wave equation $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$

$A=c^2$, $B=0$, $C=-1$

$B^2-4AC=0-4*c^2*(-1)=4c^2>0$

$\therefore$ The equation is hyperbolic

2- The heat equation $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

$A=c^2$, $B=0$, $C=0$

$B^2-4AC=0-4*c^2*0=0$

$\therefore$ The equation is parabolic

3- The two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$A=1\,,\ B=0\,,\ C=1$

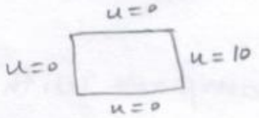
$B^2-4AC = 0-4*1*1 = -4 < 0$

$\therefore$ The equation is elliptic

Types of B.C's

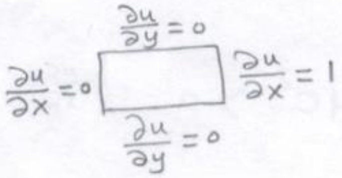
1- The Dirichlet problem if u is prescribed on the boundary surface. الدالة معرفة على الحدود

مثال



2- The Neumann problem if the normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on the boundary surface. المشتقة معرفة على الحدود

مثال



3- The mixed problem if u is prescribed on a portion of the boundary surface and u_n on the rest of it. خليط من النوعين اعلاه

Fundamental Theorem 1 (Superposition principle)

If u_1 and u_2 are any solutions of a linear homogeneous PDE in some region, then

$$u = c_1u_1 + c_2u_2$$

where c_1 and c_2 are any constants, is also a solution of that equation in that region.

proof. substitute in Eq. (3) $\quad \frac{\partial^2(c_1u_1+c_2u_2)}{\partial x^2}$

$$c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2} + c_1 \frac{\partial^2 u_1}{\partial y^2} + c_2 \frac{\partial^2 u_2}{\partial y^2} = 0$$

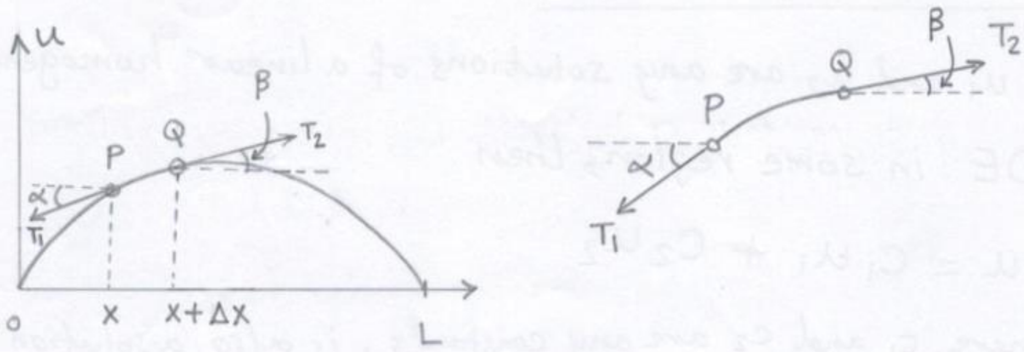
$$c_1 \left[\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right] + c_2 \left[\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} \right] = 0$$

since u_1 & u_2 are solutions, then the expressions in brackets is zero, and the statement of the theorem is proved.

11.2 One-Dimensional Wave Equation

The equation of vibrating string

assume that an elastic string is stretched to length L and fixed at the endpoints. If the string is distorted (شوهت) and then at a certain instant $(t=0)$ is released and allowed to vibrate. Find its deflection $u(x,t)$ at any x and at any $t>0$.



vibrating string

<u>Assumptions</u>

1- The mass per unit length is constant (homogeneous string).

2- The tension before fixing it is so large, so the gravitational force is neglected.

3- The string performs a small transverse motion in a vertical plane, so the deflection and the slope at every point of the string remain small

Let T_1 and T_2 be the tension at the endpoints P and Q of a small portion of the string

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{Constant} \quad \text{—(1)}$$

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \, \Delta X \frac{\partial^2 u}{\partial t^2} \quad \text{—(2)}$$

صيغة قانون نيوتن الثاني

ρ = mass of the undeflected string per unit length.

ΔX = is the length of the portion of the undeflected string.

$\frac{\partial^2 u}{\partial t^2}$ = acceleration, evaluated at some point between x and $X + \Delta X$

By using (1) we obtain

لو قسمنا معادلة ٢ على معادلة ١

$$\frac{\cancel{T_2} \sin\beta}{\cancel{T_2} \cos\beta} - \frac{\cancel{T_1} \sin\alpha}{\cancel{T_1} \cos\alpha} = \frac{\rho \, \Delta X}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho \, \Delta X}{T} \frac{\partial^2 u}{\partial t^2}$$

الميل = الظل

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_X \quad \text{and} \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{X+\Delta X}$$

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

If $\Delta x \to 0$ we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

let $c = \sqrt{\frac{T}{\rho}}$ = wave velocity $\left(\frac{m}{s}\right)$

$$\therefore \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$$

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	**Lecture sequences: 7**	**Seventh lecture**	**Instructor Name: Dr. Saddam Atteyia**
	The major contents: **1- Method of Separating Variables**		
	The detailed contents:		

11.3 Method of Separating Variables (Product Method)

The vibrations of an elastic string, are governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -(1)$$

$u(x,t)$ is the deflection of the string.

Since the string is fixed at the ends $x=0$ and $x=L$, we have 2 boundary conditions

$$u(0,t)=0 \ , \ u(L,t)=0 \qquad -(2)$$

The form of the motion of the string depend on the deflection at $t=0$ and on the velocity at $t=0$.

$\therefore$ The 2 initial conditions

$$u(x,0) = f(x) \qquad -(3)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \qquad -(4)$$

We shall proceed step by step

1. we shall obtain 2 ODE
2. we shall determine solutions of those 2 ODE that satisfy the B.C's.
3. Composed those solutions to obtaine a solution

of equation (1) that satisfy the I.C's

First step

Let $\boxed{u(x,t) = F(x)\,G(t)}$ —(5)

$$\frac{\partial^2 u}{\partial t^2} = F\frac{d^2G}{dt^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2F}{dx^2}G$$

substitute in eq. (1) we get

$$F\frac{d^2G}{dt^2} = c^2\frac{d^2F}{dx^2}G$$

Dividing by $c^2F(x)\,G(t)$

$$\frac{1}{c^2}\cdot\frac{1}{G(t)}\cdot\frac{d^2G}{dt^2} = \frac{1}{F(x)}\cdot\frac{d^2F}{dx^2} = \text{constant} = K$$

$$\boxed{F'' - KF = 0} \quad —(6)$$

$$\boxed{\ddot{G} - c^2KG = 0} \quad —(7)$$

Second Step.

determine solutions F and G so that u satisfies the B.C's

$u(0,t) = F(0)\,G(t) = 0$

$u(L,t) = F(L)\,G(t) = 0$

If $G(t) = 0$, then $u(x,t) = 0$ (which is of no interest)

$\therefore\ G(t) \neq 0$

$\therefore \quad F(0) = 0 \quad \& \quad F(L) = 0 \quad \text{—— (8)}$

For $K = 0$ from eq. (6)

$$F = ax + b \qquad \begin{array}{l} 0 = a(0) + b \Rightarrow b = 0 \\ 0 = a(L) \Rightarrow = a = 0 \end{array}$$

from eq. (8) $\quad a = b = 0 \Rightarrow F = 0 \Rightarrow u = 0$
which is of no interest.

For positive $K = \mu^2$ the general solution of eq. (6) is

$$F = A e^{\mu x} + B e^{-\mu x}$$

from eq. (8)

$$0 = A + B \Rightarrow A = -B$$

$$0 = A e^{\mu L} + B e^{-\mu L} \Rightarrow 0 = -B e^{\mu L} + B e^{-\mu L}$$

$$0 = B(e^{-\mu L} - e^{\mu L})$$

$$\therefore B = 0$$

$$\therefore A = 0$$

we obtain $F = 0 \Rightarrow u = 0$ (no solution)

For K negative $(K = -P^2)$, then from eq. (6)

$$F'' + P^2 F = 0$$

Its general solution is

$$F(x) = A \cos px + B \sin px$$

From eq. (8) we have

$$0 = A\overset{1}{\cos 0} + B\overset{0}{\sin 0}$$

$$\therefore A = 0$$

$$0 = B\sin PL$$

$$B \neq 0$$

$$\therefore \sin PL = 0$$

$$PL = n\pi$$

$$P = \frac{n\pi}{L} \quad (n \text{ integer}) \qquad -(9)$$

$$\therefore F(x) = B\sin\frac{n\pi}{L}x$$

, If we setting $B = 1$ هذه تعني انه يمكن الحصول على ∞ من الحلول

$$\therefore F(x) = F_n(x)$$

$$F_n(x) = \sin\frac{n\pi}{L}x \;,\; n = 1, 2, \cdots \qquad -(10)$$

وهي تحقق معادلة (8)

Eq. (7) takes the form

$$\ddot{G} + c^2p^2G = 0$$

$$\ddot{G} + \left(\frac{cn\pi}{L}\right)^2 G = 0$$

$$\ddot{G} + \lambda_n^2 G = 0 \quad \Rightarrow \quad \lambda_n = \frac{cn\pi}{L}$$

& A general solution to it is

$$G_n(t) = C_n\cos\lambda_n t + D_n\sin\lambda_n t$$

Hence $u_n(x,t) = F_n(x)\,G_n(t)$

$$\therefore \boxed{u_n(x,t) = (C_n\cos\lambda_n t + D_n\sin\lambda_n t)\sin\frac{n\pi}{L}x}$$

$$(n = 1, 2, \cdots) \qquad -(11)$$

- $u_n(x,t)$ are called the eigenfunctions, or characteristic functions

- The values $\lambda_n = \frac{cn\pi}{L}$ are called the eigenvalues, or characteristic values

أطياف

- $\lambda_1, \lambda_2, \ldots$ is called the spectrum.

- Each u_n represents a harmonic motion having the frequency $\frac{\lambda_n}{2\pi} = \frac{cn}{2L}$ cycles per unit time.

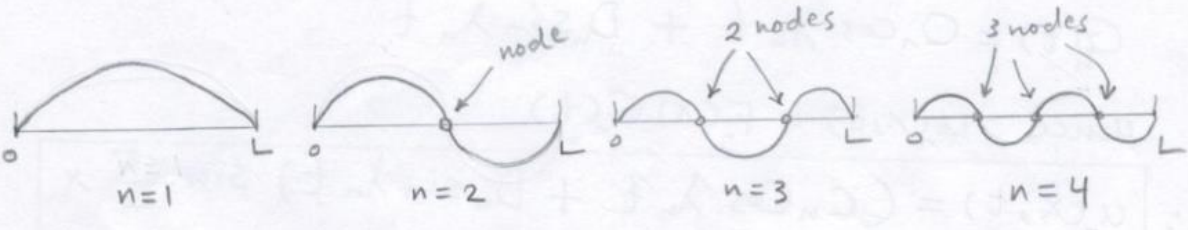
شكل

- This motion is called the nth normal mode of the string.

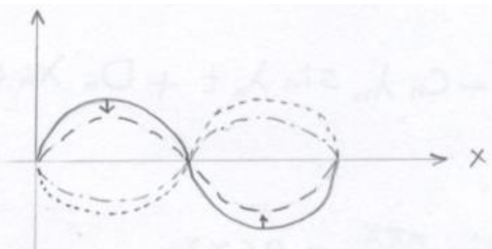
- The first normal mode is known as the fundamental mode $(n=1)$, and the others are known as overtones نغمات توافقية

عقد

- The nth normal mode has $(n-1)$ nodes (the points of the string that do not move).



Normal modes



second normal mode for various values of t

Third step

To obtain a solution that satisfies (3) and (4), we consider the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin \frac{n\pi}{L} x \qquad -(12)$$

where $\lambda_n = \frac{cn\pi}{L}$

Apply initial condition eq. (3)

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{L} x = f(x)$$

The coefficients C_n must be chosen so that $u(x,0)$ becomes a half-range expansion of $f(x)$, namely, the Fourier sine series of $f(x)$.

$$\boxed{C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx} \qquad n = 1, 2, \dots \qquad -(14)$$

similarly for the second initial condition [eq. (4)]

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-C_n \lambda_n \sin \lambda_n t + D_n \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L} \right]_{t=0}$$

$$= \sum_{n=1}^{\infty} D_n \lambda_n \sin \frac{n\pi x}{L} = g(x)$$

$$D_n \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

$$\boxed{D_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx} \qquad n=1,2,\cdots \qquad (15)$$

If the initial velocity $g(x)=0$, then $D_n=0$ and eq. (12) reduces to

$$u(x,t)=\sum_{n=1}^{\infty} C_n \cos\lambda_n t \sin\frac{n\pi x}{L} \quad , \lambda_n=\frac{cn\pi}{L}$$

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

Lecture Contents	Lecture sequences: 8	eighth lecture	Instructor Name: Dr. Saddam Atteyia
	The major contents: 1- Heat Flow 2- Laplace Equation		
	The detailed contents:		

Heat Flow

The heat flow in a body of homogeneous material is governed by the heat equation

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u \quad , \quad c^2 = \frac{K}{\sigma \rho}$$

Where $u(x,y,z,t)$ is the temperature in the body, K is the thermal conductivity, σ is the specific heat and ρ is the density of the material of the body. $\nabla^2 u$ is the Laplacian of u, and with respect to Cartesian Coordinates x, y, z,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Application

Consider the temperature in a long thin bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis (Fig. 275) and is perfectly insulated laterally, so that heat flows in the x-direction only.

Fig. 275 . Bar under consideration

Then u depends only on x and time t, and the heat equation becomes the so-called one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad —(1)$$

let the ends $x=0$ and $x=L$ of the bar are kept at temperature zero. Then the B.C's are

$$u(0,t)=0 \;,\; u(L,t)=0 \quad \text{for all } t. \quad —(2)$$

let $f(x)$ be the initial temperature in the bar. Then the initial condition is

$$u(x,0)=f(x) \quad [f(x) \text{ given}] \quad —(3)$$

we shall determine a solution $u(x,t)$ by applying the method of separation of variables.

$$u(x,t)=F(x)G(t) \quad —(4)$$

substituting this expression into (1)

$$F\dot{G} = c^2 F'' G$$

divide this equation by $c^2 FG$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} \quad —(5)$$

both expressions must be equal a constant $K=-p^2$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = K = -p^2$$

this yields the two linear ordinary differential equations

$$F'' + p^2 F = 0 \quad \text{———} (6)$$

and

$$\dot{G} + c^2 p^2 G = 0 \quad \text{———} (7)$$

For eq.(6) the general solution

$$F(x) = A \cos px + B \sin px \quad \text{———} (8)$$

From B.C's (2) it follows that

$$u(0,t) = F(0)\,G(t) = 0 \quad \text{and} \quad u(L,t) = F(L)\,G(t) = 0$$

if $G(t) = 0 \Rightarrow u = 0$

$\therefore F(0) = 0$ and $F(L) = 0$, substitute in eq.(8)

$F(0) = A \Rightarrow A = 0$ and $F(L) = B \sin pL$

$B \neq 0$

$\therefore \sin pL = 0$ hence $p = \frac{n\pi}{L}$, $n = 1, 2, \dots$

setting $B = 1$

$\therefore$ the solutions of (6)

$$F_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, \dots$$

eq.(7) takes the form

$$\dot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = \frac{cn\pi}{L}$$

The general solution is

$$G_n(t) = B_n e^{-\lambda_n^2 t} \qquad n = 1, 2, \dots$$

where B_n is a constant.

$$\therefore \boxed{u_n(x,t) = F_n(x)\, G_n(t) = B_n \sin\frac{n\pi x}{L}\, e^{-\lambda_n^2 t}} \quad n=1,2,\dots \qquad (9)$$

To obtain a solution also satisfy I.C, we consider the series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-\lambda_n^2 t} \qquad (10)$$

$$\left(\lambda_n = \frac{cn\pi}{L}\right)$$

Apply eq. (3)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$$

the coefficients B_n must be chosen such that $u(x,0)$ becomes a half-range expansion of $f(x)$, namely, the Fourier sine series of $f(x)$

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx} \qquad n=1,2,\cdots \qquad (11)$$

Because of the exponential factor all the terms in (10) approach zero as t approaches infinity. The rate of decay increases with n.

13. (Insulated ends, adiabatic boundary conditions)
Find $u(x,t)$ in a bar of length L that is perfectly

insulated, also at the ends at $x=0$ and $x=L$, assuming that $u(x,0)=f(x)$. physical information: the flux of heat through the faces at the ends is proportional to the values of $\partial u/\partial x$ there. Show that this situation corresponds to the conditions

$$u_x(0,t)=0, \quad u_x(L,t)=0, \quad u(x,0)=f(x)$$

Show that the method of separating variables yields the solution

$$\overset{0}{q} = -KA\frac{\partial T}{\partial x}$$

$$\therefore \frac{\partial T}{\partial x}=0$$

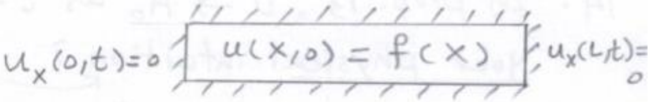
هنا $T=u$

$$u(x,t)=A_0+\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-(cn\pi/L)^2 t}$$

where

$$A_0=\frac{1}{L}\int_0^L f(x)\,dx, \quad A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n=1,2,\dots$$

solution:



$$F''+p^2F=0$$

$$\dot{G}+c^2p^2G=0$$

$$F(x)=A\cos px+B\sin px$$

Apply B.C's

$$F'(x)=-A\sin px\cdot p+B\cos px\, p$$

$$0=0+Bp \Rightarrow B=0$$

$$0=-Ap\sin pL$$

$$\sin pL=0 \Rightarrow pL=n\pi \Rightarrow p=\frac{n\pi}{L}, \quad \text{setting } A=1$$

$$F_n(x)=\cos\frac{n\pi x}{L}$$

$$G_n(t)=A_n e^{-\lambda_n^2 t}$$

$$u(x,t)=\sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{L}\, e^{-\lambda_n^2 t}$$

$$=A_0+\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\, e^{-\lambda_n^2 t}$$

Apply I.C $u(x,0) = f(x)$

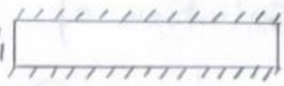
$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$

It is a half range Fourier cosine series with

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad , n=1,2,\dots$$

11. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

$u(0,t)=U_1$  $u(L,t)=U_2$

let $u(x,t) = u_I(x) + u_{II}(x,t)$

$$\frac{\partial u}{\partial t} = 0 + \frac{\partial u_{II}}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 u_I}{dx^2} + \frac{\partial^2 u_{II}}{\partial x^2}$$

$$\frac{1}{c^2}\frac{\partial u_{II}}{\partial t} = \frac{\partial^2 u_{II}}{\partial x^2} + \frac{d^2 u_I}{dx^2}$$

$$u(0,t) = u_I(0) + u_{II}(0,t)$$

$$U_1 = U_1 + u_{II}(0,t)$$

$$u_{II}(0,t) = 0$$

$$u(L,t) = u_I(L) + u_{II}(L,t)$$

$$U_2 = U_2 + u_{II}(L,t)$$

$$u_{II}(L,t) = 0$$

$$\left.\begin{aligned} &\frac{1}{c^2}\frac{\partial u_{II}}{\partial t} = \frac{\partial^2 u_{II}}{\partial x^2} \\ &u_{II}(0,t) = 0 \\ &u_{II}(L,t) = 0 \end{aligned}\right\} \text{ exist only when } \frac{d^2 u_I}{dx^2} = 0$$

$$\frac{d^2u_1}{dx^2}=0 \Rightarrow \frac{du_1}{dx}=A \Rightarrow u_1(x)=Ax+B$$

$$u_1(0)=U_1=B$$

$$u_1(L)=AL+B$$

$$U_2=AL+U_1$$

$$A=\frac{U_2-U_1}{L}$$

$$\therefore\ u_1(x)=\frac{U_2-U_1}{L}x+U_1$$

The solution of

$$\frac{\partial u_{II}}{\partial t}=c^2\frac{\partial^2 u_{II}}{\partial x^2}$$

$$u_{II}(0,t)=0$$

$$u_{II}(L,t)=0$$

is obtained previosuly

$$u_{II}(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}e^{-\lambda_n^2 t}$$

when $t\to\infty \Rightarrow e^{-\lambda_n^2 t}=0 \Rightarrow u_{II}(x,t)=0$

$$\therefore u(x,t)=u_1(x)$$

12.

$$u(x,t)=u_1(x)+u_{II}(x,t)$$

$$u(x,0)=u_1(x)+u_{II}(x,0)$$

$$u_{II}(x,0)=f(x)-u_1(x)$$

$$u_{11}(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$f(x) - u_1(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_1(x)) \sin\frac{n\pi x}{L}\, dx$$

Laplace Equation

One of the most important partial differential equations in physics is Laplace's equation

$$\boxed{\nabla^2 u = 0}$$

Here $\nabla^2 u$ is the Laplacian of u. In Cartesian coordinates x, y, z in space,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

For the two dimensional case, when u depends on two variables only the above equation reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

This equation will be considered in some region R of the xy-plane and a given boundary condition on the boundary curve of R. This is called a boundary value problem.

Let us consider a problem in a rectangle R (Fig. 1), assuming that the temperature $u(x,y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle R.

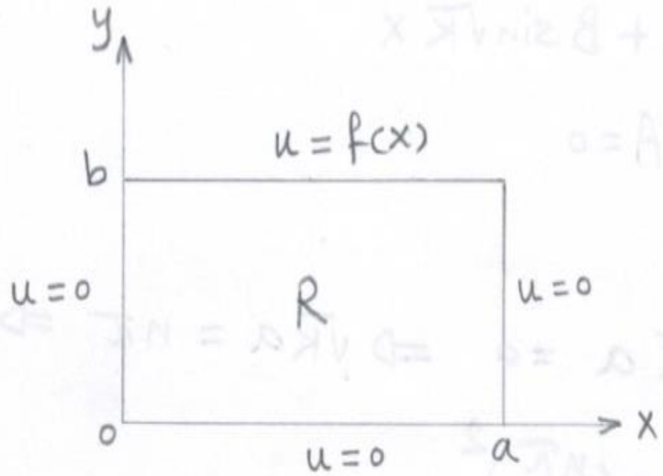


Fig 1. Rectangle R and given boundary values

we solve this problem by separating variables.
substituting of

$$u(x,y) = F(x)\, G(y)$$

into (1)

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2F}{dx^2} G(y) \quad , \quad \frac{\partial^2 u}{\partial y^2} = F(x) \frac{d^2G}{dy^2}$$

$$\frac{d^2F}{dx^2} G(y) + F(x) \frac{d^2G}{dy^2} = 0$$

division by FG gives

$$\frac{1}{F}\cdot\frac{d^2F}{dx^2} = -\frac{1}{G}\cdot\frac{d^2G}{dy^2} = -K$$

$$\therefore \frac{d^2F}{dx^2} + KF = 0$$

& $F(0) = 0 \quad , \quad F(a) = 0$

$$F(x) = A\cos\sqrt{K}\,x + B\sin\sqrt{K}\,x$$

$F(0) = 0 = A \Rightarrow A = 0$

$F(a) = 0 = B\sin\sqrt{K}\,a$

$B \neq 0 \Rightarrow \sin\sqrt{K}\,a = 0 \Rightarrow \sqrt{K}\,a = n\pi \Rightarrow$

$\sqrt{K} = \frac{n\pi}{a} \Rightarrow K = \left(\frac{n\pi}{a}\right)^2$

setting $B = 1$

$\therefore F_n(x) = \sin\frac{n\pi}{a}x \quad , n = 1, 2, \cdots$

The equation for G then becomes

$$\frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

solutions are

$$G_n(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}$$

$u(x,0) = 0 \Rightarrow G_n(0) = 0 = A_n + B_n \Rightarrow$

$$B_n = -A_n$$

$$\therefore\ G_n(y) = A_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$$= 2A_n \sinh\frac{n\pi y}{a} \qquad \left(\sinh x = \frac{e^{x}-e^{-x}}{2}\right)$$

let $2A_n = A_n^*$

$$\therefore\ u_n(x,y) = F_n(x)\,G_n(y)$$

$$= A_n^* \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi y}{a}$$

To obtain the solution of our problem also satisfying the boundary condition $u(x,b) = f(x)$ on the upper side of R, we consider the infinite series

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi y}{a}$$

Thus, at $y = b$

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi b}{a}$$

$$= \sum_{n=1}^{\infty}\left(A_n^* \sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

This shows that the expressions in the parantheses must be the Fourier Coefficients b_n of $f(x)$;

$$\therefore\ b_n = A_n^* \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

$\therefore$ The solution of our problem is

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi y}{a}}$$

where

$$\boxed{A_n^* = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx}$$

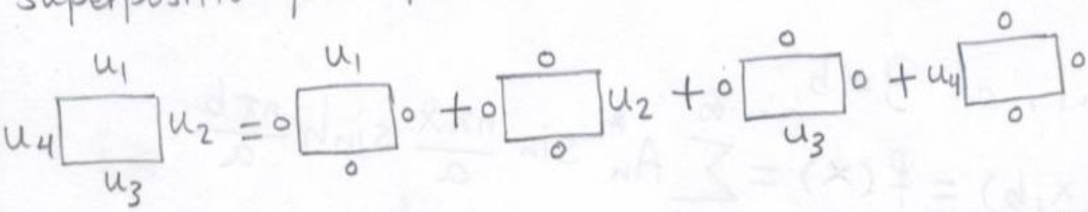
Note : superposition principle :

If u_1 & u_2 are any solutions of a linear homogeneous PDE in some region, then

$u = c_1 u_1 + c_2 u_2$

where c_1 & c_2 are any constants, is also a solution of that equation in that region.

if we have an Laplace equation with nonhomogenous B.C's & the solution need 3-homogeneous B.C's so we used superposition principle as shown below



Then the complet solution is the summation of the solutions of the Four cases (four B.C's) assumed.

Note: The two-dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

If the heat flow is steady (i.e., time independent), then $\frac{\partial u}{\partial t} = 0$, and the heat equation reduces to Laplace's equation.

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

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1- Complex Number
2- Polar Form of Complex Numbers</td></tr>
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</table>

12.1 Complex Number

$$z = (\underset{\substack{\uparrow \\ \text{real part}}}{x} , \underset{\substack{\uparrow \\ \text{imaginary part}}}{y})$$

$$z = \underset{\substack{\uparrow \\ \text{Re } z}}{x} + \underset{\substack{\uparrow \\ \sqrt{-1}}}{i}\, \underset{\substack{\uparrow \\ \text{Im } z}}{y}$$

$$i^2 = -1 \ , \ i^3 = -i \ , \ i^4 = +1$$

if $z_1 = x_1 + i y_1$ and $z_2 = x_2 + i y_2$

then

1) $z_1 = z_2$ if $x_1 = x_2$ & $y_1 = y_2$ مساواة

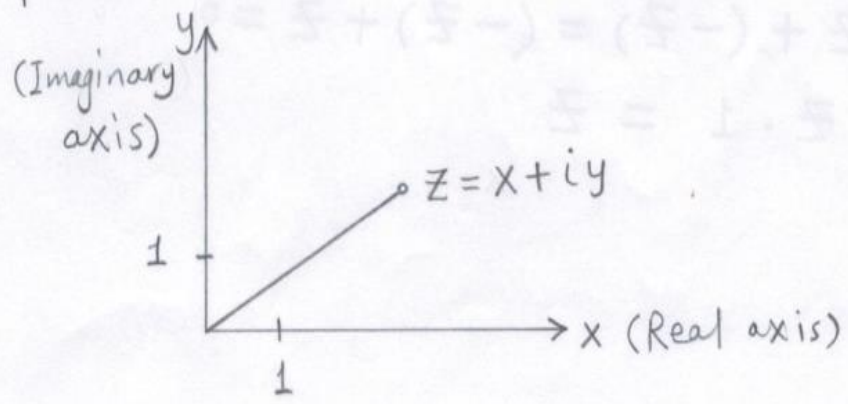
2) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$ جمع وطرح

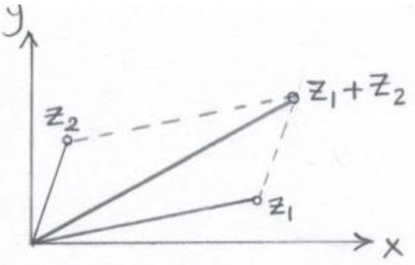
3) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ ضرب

4) $$\frac{z_1}{z_2} = \frac{x_1 + i y_1}{x_2 + i y_2} * \frac{x_2 - i y_2}{\underbrace{x_2 - i y_2}} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

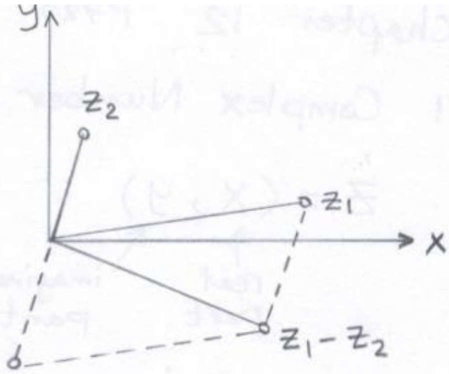
مترافق Conjugate

Complex plane





Addition of complex numbers



Subtraction of Complex numbers

properties of the Arithmetic Operations

1) $\left.\begin{aligned} z_1+z_2 &= z_2+z_1 \\ z_1 z_2 &= z_2 z_1 \end{aligned}\right\}$ التبادل (commutative laws)

2) $\left.\begin{aligned} (z_1+z_2)+z_3 &= z_1+(z_2+z_3) \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3) \end{aligned}\right\}$ الترتيب (Associative laws)

3) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$ التوزيعي (Distributive law)

4) $0+z = z+0 = z$

$z+(-z) = (-z)+z = 0$

$z \cdot 1 = z$

Complex conjugate Numbers

$z = x+iy$ conjugate $\bar{z} = x-iy$

12.2 Polar Form of Complex Numbers. Powers and Roots

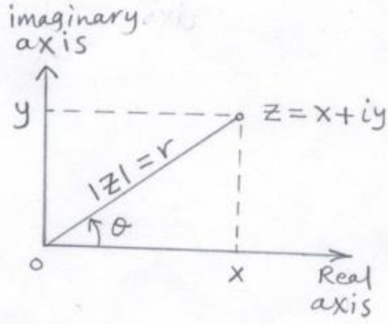
$z = x + iy$

in terms of polar coordinates r, θ

$x = r\cos\theta$, $y = r\sin\theta$

By substituting this we obtain

$z = r\cos\theta + i r\sin\theta$
$= r(\cos\theta + i\sin\theta)$



Complex plane, Polar form of a complex number

r is called absolute value or modulus of z ($|z|$).

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

Geometrically, $|z|$ is the distance of the point z from the origin.

θ is called the argument (الإزاحة الزاوية) of z and is denoted by arg z.

$$\theta = \arg z = \arctan\frac{y}{x}$$

Geometrically, θ is the directed angle from the positive x-axis to OZ and measured in radians and positive in the counterclockwise sense (اتجاه).

working with conjugates is easy, since we have

$$\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$$

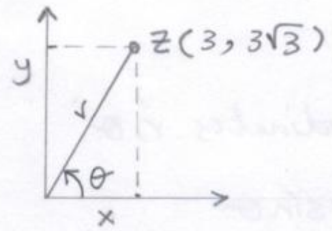
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Ex: Let $z = 3 + 3\sqrt{3}\, i$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9+9*3} = 6$$

$$\theta = \arg z = \tan^{-1}\frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

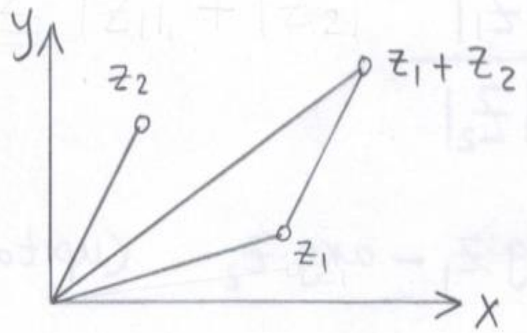
$$z = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \quad \text{polar form}$$



Triangle inequality

For any complex numbers z_1 and z_2

$$|z_1 + z_2| \leqq |z_1| + |z_2|$$



Triangle inequality

the triangle inequality can be extended to arbitrary sums.

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|$$

EX: if $z_1 = 1 + i$ and $z_2 = -2 + 3i$

$$|z_1 + z_2| = |-1+4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.019$$

Multiplication and Division in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)]$$

$$\therefore\ z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$|z_1 z_2| = |z_1||z_2|$$

and

مضاعفات

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad \text{(up to multiples of } 2\pi)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \quad \text{(up to multiples of } 2\pi)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

EX: Let $z_1 = -2 + 2i$ and $z_2 = 3i$

Then $z_1 z_2 = -6i - 6 = -6 - 6i$

$$\frac{z_1}{z_2} = \frac{-2+2i}{3i} * \frac{-3i}{-3i} = \frac{6i+6}{9} = \frac{2}{3} + \frac{2}{3}i$$

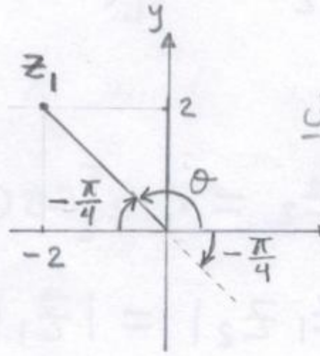
$$|z_1 z_2| = |z_1||z_2|$$

$$|z_1| = \sqrt{4+4} = \sqrt{8}$$

$$|z_2| = \sqrt{9}$$

$$|z_1 z_2| = \sqrt{8}\,\sqrt{9} = 6\sqrt{2}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{2\sqrt{2}}{3}$$



$$\operatorname{Arg} z_1 = \tan^{-1}\frac{2}{-2} = -\frac{\pi}{4}$$

$$\theta = \left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}$$

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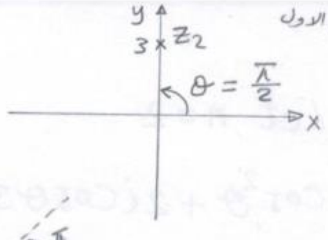
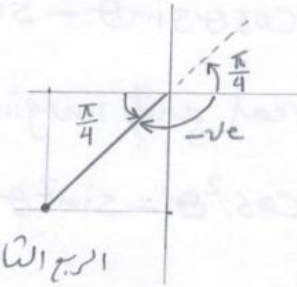


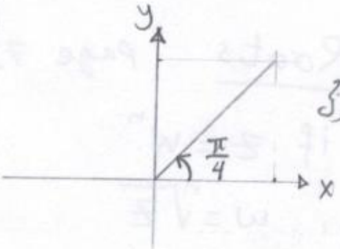
$$\operatorname{Arg} z_2 = \tan^{-1}\frac{3}{0} = \frac{\pi}{2}$$

$$\operatorname{Arg} z_1 z_2 = \tan^{-1}\frac{-6}{-6} = \frac{\pi}{4}$$

$$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \quad +ve$$

$$\theta = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4} \quad -ve$$





$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \tan^{-1}\frac{\frac{2}{3}}{\frac{2}{3}} = \frac{\pi}{4}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$
$$= \frac{3\pi}{4} + \frac{\pi}{2} = \frac{3\pi + 2\pi}{4} = \frac{5\pi}{4} \quad +ve$$ نفس النتيجة السابقة

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$
$$= \frac{3\pi}{4} - \frac{\pi}{2} = \frac{3\pi - 2\pi}{4} = \frac{\pi}{4}$$

Integer powers of z

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^{-2} = r^{-2}\left[\cos(-2\theta) + i\sin(-2\theta)\right]$$

generally for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

if $r=1$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$ formula of De Moivre

Let $n=2$

$$\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta$$

take the real and imaginary parts on both sides

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad , \quad \sin 2\theta = 2\cos\theta\sin\theta$$

Roots page 730

if $z = w^n \qquad (n=1,2,\cdots) \quad (z \neq 0)$

$w = \sqrt[n]{z}$

$$w = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right) \quad k=0,1,\cdots,n-1.$$

These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute (تكون) the vertices of a regular polygon of n sides.

The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k=0$ in the above equation is called the principal value of $w=\sqrt[n]{z}$.

Lectures of the Department of Mechanical Engineering

Subject Title: Engineering Analysis

Class: Third Class

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Dr. Saddam Atteyia</td></tr>
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1- Curves and Regions in the Complex Plane
2- Limit, Derivative, Analytic Function</td></tr>
<tr><td colspan="3">The detailed contents:</td></tr>
</table>

12.3 Curves and Regions in the complex plane

The distance between two points z and a is $|z-a|$. Hence a circle C of radius ρ and center at a Fig.(1) can be represented by

$$|z-a| = \rho \qquad —(1)$$

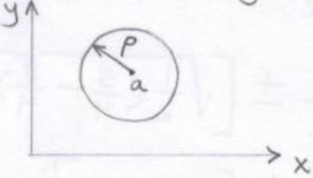
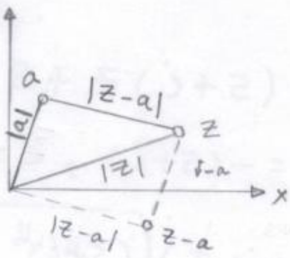


Fig.(1)

توضيح

distance between 2 points in the complex plane



the unit circle is the circle of radius 1 and center at the origin $a=0$ (Fig 2), is given by

$$|z| = 1$$

$$\text{radius} = 1$$

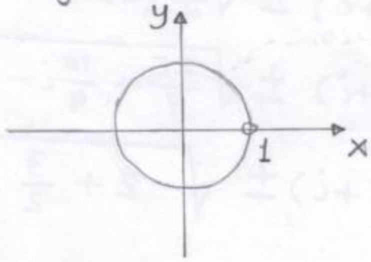
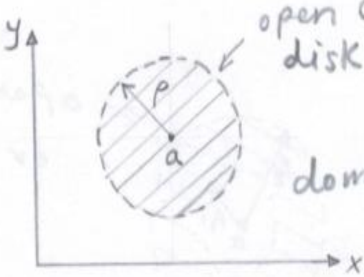


Fig 2
Unit circle

the inequality

$$|z-a| < \rho \qquad —(2)$$

this represents the interior of a circle C. Such a region called a circular disk



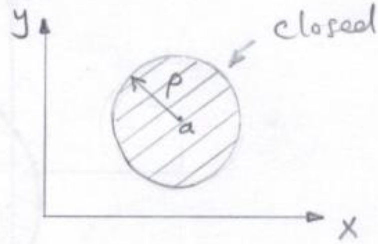
open circular disk

قرص داخل دائرة

domain

$|z-a| \leqq \rho$ القرص والدائرة

consists of the interior of C and C itself

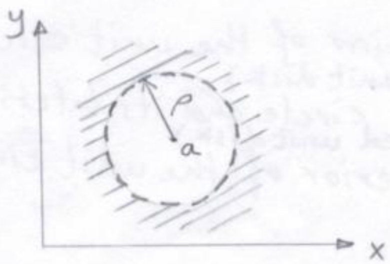


region

Similarly, the inequality

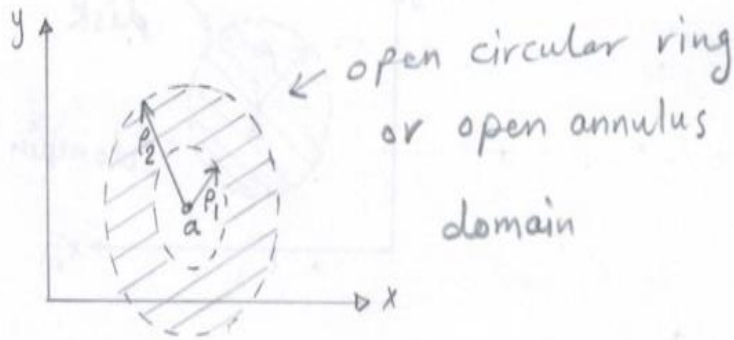
$|z-a| > \rho$

represents the exterior of the circle C.



the region between two concentric circles of radii ρ_1 and ρ_2 $(\rho_2 > \rho_1)$ can be represented in the form

$$\rho_1 < |z-a| < \rho_2 \qquad (3)$$



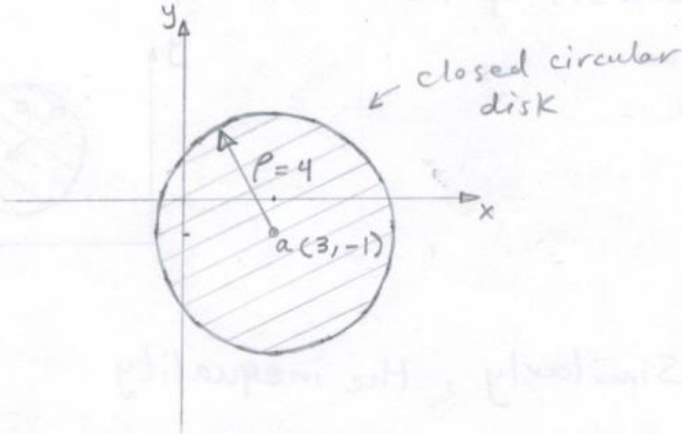
domain

EX 1 : Determine the region in the complex plane given by

$|z-3+i| \leqq 4$

$a = 3-i$

$\rho = 4$



EX2 : Determine each of the regions

(a) $|z|<1$ (b) $|z| \leqq 1$ (c) $|z|>1$ $(a=0)$

a هي مركز الدائرة وهي هنا نقطة الاصل

solution) (a) The interior of the unit circle. (open unit disk)

(b) The unit circle and its interior. (closed unit disk)

(c) The exterior of the unit circle.

Some Concepts Related to sets in the complex plane

* set of points (S): any sort (نوع، ترتيب) of collection (مجموعة) of finitely or infinitely many points. (EX. the solutions of a quadratic equation, the points on a line, and the points in the interior of a circle are sets.

* S is called open if every point of S has a neighborhood (جوار) consisting entirely of points that belong (تنتمي) to S. (EX. the points in the interior of a circle or a square).

* Open S is said to be connected if any two of its points can be joined by a broken line of

finitely many straight line segments all of whose points belong to S.

* An open connected set is called a domain. مجال
(EX. Thus (2) and (3) are domains).

* The complement of S is defined to be the set of all points of the complex plane that do not belong to S. تكملة/تتمة

* S is called closed if its complement is open.
(EX. $|z| \leq 1$ (closed unit disk) since its complement $|z| > 1$ is open)

* A boundary point of S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S. (EX. the boundary points of an annulus are the points on the two bounding circles. If a set of S is open, then no boundary point belongs to S. If S is closed, then every boundary point belongs to S.

* A region is a set consisting of a domain plus, perhaps, some or all of its boundary points.

12.4 Limit . Derivative . Analytic Function

Complex Function (in real) $y = f(x)$

$w = f(z)$ w complex function (value of f at z)
z complex variable

Ex: $w = f(z) = z^2 + 3z$

w is complex, $w = u + iv$
real part imaginary part

w depends on $z = x + iy$

hence u becomes a real function of x and y, and so does v.

$$\therefore \quad \boxed{w = f(z) = u(x,y) + iv(x,y)}$$

Ex1. Function of a complex variable

Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the values of f at $z = 1+3i$ and $z = 2-i$.

Solution: $u = \mathrm{Re}\, f(z) = \mathrm{Re}\left[(x+iy)^2 + 3(x+iy)\right]$

$$= \mathrm{Re}\left[x^2 + 2xiy - y^2 + 3x + 3iy\right]$$

$$= x^2 - y^2 + 3x$$

and $v = 2xy + 3y$

$$f(1+3i) = (1+3i)^2 + 3(1+3i)$$
$$= 1 + 6i - 9 + 3 + 9i$$

$= -5 + 15i$

$u = 1^2 - 3^2 + 3*1 = 1 - 9 + 3 = -5$

$v = 2*1*3 + 3*3 = 6 + 9 = 15$

$f(2-i) = (2-i)^2 + 3(2-i)$

$= 4 - 4i - 1 + 6 - 3i = 9 - 7i$

$u = 2^2 - (-1)^2 + 3*2 = 4 - 1 + 6 = 9$

$v = 2*2*(-1) + 3*(-1) = -4 - 3 = -7$

<u>Ex2</u>: Let $w = f(z) = 2iz + 6\bar{z}$. Find u and v and the value of f at $z = \frac{1}{2} + 4i$

<u>solution</u>: $f(z) = 2i(x+iy) + 6(x-iy)$

$= 2ix - 2y + 6x - 6iy$

$\therefore u = -2y + 6x$

$v = 2x - 6y$

$f(\frac{1}{2} + 4i) = 2i(\frac{1}{2} + i4) + 6(\frac{1}{2} - i4)$

$= i - 8 + 3 - 24i$

$= -5 - 23i$

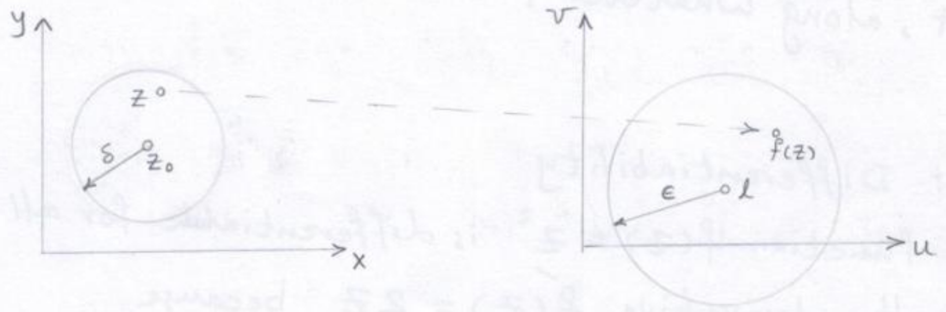
$u = -2*4 + 6*\frac{1}{2} = -8 + 3 = -5$

$v = 2*\frac{1}{2} - 6*4 = 1 - 24 = -23$

<u>Limit, Continuity</u>

1) $\lim_{z \to z_0} f(z) = \ell$

if f is defined in a neighborhood of z_0 (except at z_0 itself) and if the values of f are "close" to ℓ for all z "close" to z_0.



Limit

الـ limit يظهر اذا كانت الدالة معرفة في نقطة مجاورة لـ z_0 واذا كانت قيم f قريبة لـ ℓ لكل قيم z القريبة من z_0.

2) $\lim_{z \to z_0} f(z) = f(z_0) \Rightarrow f(z)$ is continuous

$f(z)$ is said to be continuous in a domain if it is continuous at each point of this domain.

Derivative

The derivative of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If $\Delta z = z - z_0 \Rightarrow z = z_0 + \Delta z$

$$\therefore f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

The definition of a limit implies that $f(z)$ is defined (at least) in a neighborhood of z_0. And z may approach z_0 from any direction. Hence differentiability at z_0 means that, along whatever path z approaches z_0.

Ex3: Differentiability

The function $f(z) = z^2$ is differentiable for all z and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z\Delta z + \Delta z^2 - \cancel{z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\frac{2z\cancel{\Delta z}}{\cancel{\Delta z}} + \frac{\Delta z^{\cancel{2}}}{\cancel{\Delta z}} \right)$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

The differentiation rules are the same as in the real calculus, Thus

$$(cf)' = cf', \quad (f+g)' = f' + g', \quad (fg)' = f'g + fg',$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$(z^n)' = n z^{n-1}$ (n integer)

- if $f(z)$ is differentiable at z_0, it is continuous at z_0

<u>EX 4:</u> $\bar{z}$ not differentiable

$f(z) = \bar{z} = x - iy$

we write $\Delta z = \Delta x + i \Delta y$

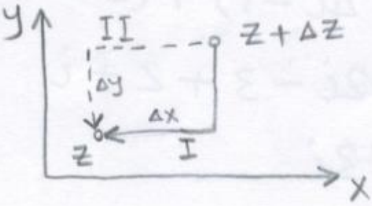
we have

$$(*) \quad \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\Delta \bar{z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0 \rightarrow +1$ (path I)

If $\Delta x = 0 \rightarrow -1$ (path II)

Hence the above equation(*) approaches +1 along path I, but −1 along path II



Hence the limit of the above equation as $\Delta z \to 0$ does not exist at any z.

<u>Analytic Functions</u>

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighbohood of z_0.

Before we consider special analytic functions (exponential functions, cosine, sine, etc.). let us give equations by means of which we can readily decide whether a function is analytic or not. These are the famous Cauchy-Riemann equations.