محاضرات قسم الهندسة الميكانيكية

Subject Title: Strength of Materials
Class: Second Year
Instructor Name: Dr. Ziad Sh. AL Sarraf

Lecture 1	**Introduction- Strength of Materials.**
	Introduction; syllabus; Classification of engineering mechanics; Definition of mechanics of materials; Why do we study mechanics of materials; classification of materials properties; Mechanical properties of the materials; Tensile test; Hardness; Impact test; creeping; Fatigue and Fatigue limit.
Lecture 2&3	**Simple Stresses and Strains. Thermal Stresses.**
	Introduction; Normal (Direct) stress; Direct strain; Sign convention for direct stress and strain; Bearing stress; Elastic materials Hook's law); Modulus of elasticity (Young Modulus); Tensile test; Ductile and Brittle materials; Poisson's ratio; Application of Poisson's ratio to a two dimensional stress system; Shear stress and shear strain; Allowable Working Stress – Factor of Safety; Temperature stress.
Lecture 4&5	**Compound Bars.**
	Introduction; Statically determinate and statically indeterminate systems; Compound bar subjected to external load; Equivalent or combined modulus; Compound bar subjected to temperature change; Problems.
Lecture 6&7	**Shearing Force and Bending Moment Diagrams.**
	Introduction; What is a beam; Beam types; Shear force and bending moment definitions; shear force and bending moment sign conventions; Types of loading beams; S.F. and B.M. diagrams for beams carrying concentrated loads only; S.F. and B.M. diagrams for uniformly distributed loads (u.d.l.); S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads (u.d.l.); Point of Contraflexure; Relationship between S.F. and B.M. and intensity of loading; S.F. and B.M. diagrams for an applied couple or moment; and for inclined loads; Problems.
Lecture 8&9	**Bending.**
	Introduction; Simple bending theory; Neutral axis; Section modulus; Second moment of area; Skew loading; Bending of composite or flitched beams; Combined bending and direct stress -eccentric loading; "Middle-quarter " and "middle-third " rules; Shear stresses owing to bending; Strain energy in bending; Limitations of the simple bending theory; "Middle-quarter " and "middle-third " rules; Problems.

Lecture 10&11	**Slope and deflection of beams.**
	Introduction; Direct integration method; Macaulay's method; Macaulay's method for u.d.l; Macaulay's method for beams with u.d.1. applied over part of the beam; Macaulay's method for couple applied at a point; Mohr's "area-moment" method; Principle of superposition; Energy method; Relationship between loading, S.F., B.M., slope and deflection; Problems.
Lecture 12&13	**Shear stress distribution.**
	Introduction; Distribution of shear stress due to bending; Application to rectangular sections; Application to I-section beams; Vertical shear in the web; Vertical shear in the flanges; Horizontal she& in the flanges; Application to circular sections; Limitation of shear stress distribution theory; Shear centre; Problems.
Lecture 14	**Torsion.**
	Introduction; Simple torsion theory; ; Polar second moment of area; Shear stress and shear strain in shafts; Section modulus; Torsional rigidity; Torsion of hollow shafts; Torsion of thin-walled tubes; Composite shafts -series connection; Composite shafts -parallel connection; Principal stresses; Strain energy in torsion; Variation of data along shaft length -torsion of tapered shafts; Power transmitted by shafts; Problems.

Lecture 1&2&3 Feb 2018	**Thin Cylinders and Shells.**
	Introduction; Thin cylinders under internal pressure; Hoop or circumferential stress; Longitudinal stress; Changes in dimensions; Thin rotating ring or cylinder; Thin spherical shell under internal pressure; Change in internal volume; Vessels subjected to fluid pressure; Cylindrical vessel with hemispherical end; Effects of end plates and joints; Problems.
Lecture 4&5&6	**Strain Energy.**
	Introduction; Strain energy –shear; Strain energy –bending; Strain energy – torsion; Problems. Double integration method, Moment – area method, Castigliano's first theorem, Three – moment equation.
Lecture 7&8&9	**Complex Stresses.**
	Introduction; Stresses on oblique planes ; Material subjected to pure shear; Material subjected to two mutually perpendicular direct stresses; Material subjected to combined direct and shear stresses; Principal plane inclination in terms of the associated principal stress ; Graphical solution - Mohr 's stress circle. Problems.
Lecture 10&11&12	**Complex Strain and the Elastic Strain.**
	Introduction; Application of Mohr's circle to combined loadings, Mohr's circle for strains, columns, Euler's formula for long columns.
Lecture 13&14	**Flanged bolt couplings**

Introduction to Mechanics of Materials

Definition: Mechanics of materials is a branch of applied mechanics that deals with the behaviour of solid bodies subjected to various types of loading.

Why do we study Mechanics of Materials?
Anyone concerned with the strength and physical performance of natural/manmade structures should study Mechanics of Materials

Classification of Materials Properties

In engineering materials, there are several properties of the materials or metals, as mentioned below in figure:-

Mechanical properties of Materials

The properties of materials that determines its behaviour under applied loads or forces are called (Mechanical Properties) such as: [strength, hardness, elasticity, plasticity, toughness, brittleness, ductility, malleability, durability and stability], are used as measures of how materials behave under a load. They are usually related to the elastic and plastic behaviour of the material. A sound knowledge of mechanical properties of materials provides the basis for predicting behaviour of materials under different load conditions and designing the components out of them.

-Strength A property that enables a material or metal to resist deformation under load.

-Fatigue strength is the ability of material to resist various kinds of rapidly changing stresses and is expressed by the magnitude of alternating stress for a specified number of cycles.

-Impact strength is the ability of a metal to resist suddenly applied loads and is measured in foot-pounds of force.

-Hardness is the property of a material to resist indentation and scratching.
Because there are several methods of measuring hardness, the hardness of a material is always specified in terms of the particular test that was used to measure this property. Rockwell, Vickers, or Brinell are some of the methods of testing.

-Elasticity is the ability of a material to return to its original shape after the load is removed. Theoretically, the elastic limit of a material is the limit to which a material can be loaded and still recover its original shape after the load is removed.

-Plasticity is the ability of a material to deform permanently without breaking or rupturing. This property is the opposite of strength. The combination of plasticity and strength is used to manufacture large structural members. loaded and still recover its original shape after the load is removed.
-Toughness is the property that enables a material to withstand shock and to be deformed without rupturing. Toughness may be considered as a combination of strength and plasticity. When a material has a load applied to it, the load causes the material to deform. Good examples are Cast iron, concrete, high carbon steels, ceramics, and some polymers.

-Brittleness is the opposite of the property of plasticity. A brittle metal is one that breaks or shatters before it deforms. White cast iron and glass are good examples of brittle materials.

-Ductility is the property that enables a material to stretch, bend, or twist without cracking or breaking. This property makes it possible for a material to be drawn out into a thin wire.

-Malleability is the property that enables a material to deform by compressive forces without developing defects. A malleable material is one that can be stamped, hammered, forged, pressed, or rolled into thin sheets. -Durability A general property. The ability to withstand wear and tear through weathering and corrosive attack etc.

Another definition: is the ability of a product to perform its required function over a lengthy period under normal conditions of use without excessive expenditure on maintenance or repair.

Examples of durable goods include automobile, books, household goods (home appliances, consumer electronics, furniture, tools etc.), sports equipment, jewelry, medical equipment, firearms and toys.

-Stability The state of being stable. A general property of resistance to changes in shape or size.

Impact test

The purpose of impact testing is to measure an object's ability to resist high-rate loading. It is usually thought of in terms of two objects striking each other at high relative speeds. A part, or material's ability to resist impact often is one of the determining factors in the service life of a part, or in the suitability of a designated material for a particular application.

-Measure toughness of materials in terms of energy absorption.

-Specimen is impacted by a hammer and the energy absorbed during fracture is measured in Joule.

-Easy and practical.

-Establish Ductileto Brittle Transition Temperature.

Creeping

The Creep is a phenomenon where some materials grow longer over a period of time, when a constant tensile stress is applied to it. The material may well fail although the tensile stress is well below the ultimate value.

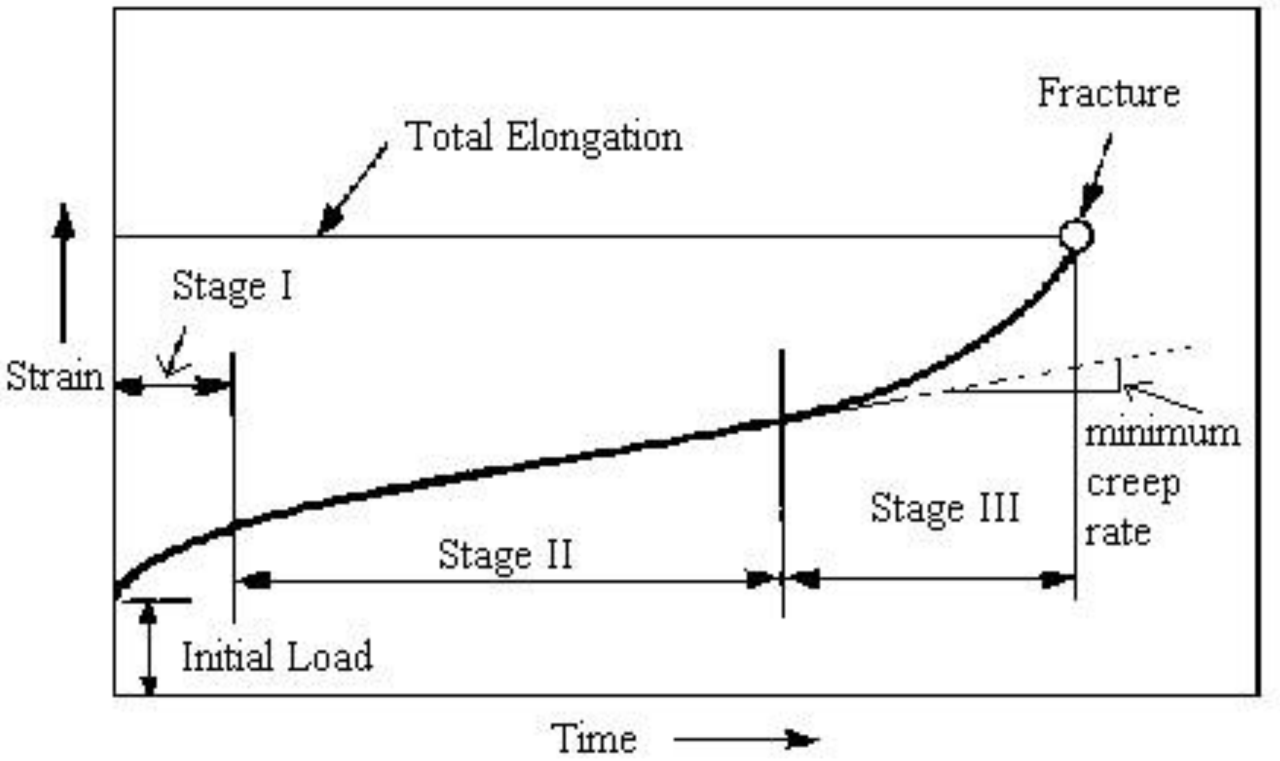
-Creep is high temperature progressive deformation of a material at constant stress.

-A tensile specimen is loaded at a constant (elevated) temperature. Strain is measured with time.

Creeping

Creep usually occurs in three stages called primary, secondary and tertiary. In the primary stage extension is fast but this stage is not always present. In the second stage the extension is at a constant rate and relatively slow. In the tertiary stage the extension quickens again and leads to failure.

The creep rate is affected by the stress. The higher the stress, the quicker the creep.



Fatigue

Fatigue is a phenomenon that occurs in a material that is subject to a cyclic stress. The repeated application of stress typically produced by an oscillating load such as vibration. Sources of ship vibration are engine, propeller and waves. Maximum stress decreases as the number of loading cycles increases.

S – N graph

Test data is presented on a S - N graph. S stands for stress and N for the number of cycles

Fatigue

- Fatigue limit, Endurance limit or fatigue strength

Are all expressions used to describe a property of materials The lower limit is called the endurance limit. If the stress level is below this limit, it will never fail. Non-ferrous materials have no endurance limit, such as Aluminium.

The diagram shows approximate fatigue characteristics of three materials. Research shows that for ferrous materials the endurance limit is approximately proportional to the tensile strength.

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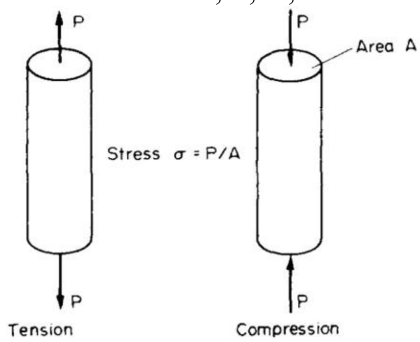
Simple Stresses and Strains. Thermal Stresses.

Chapter 1 Simple stress and Strain

If a cylindrical bar is subjected to a direct pull or push along its axis as shown, then it is said to be subjected to tension or compression. Typical examples of tension are the forces present in towing ropes or lifting hoists, whilst compression occurs in the legs of your chair as you sit on it or in the support pillars of buildings.

Force can be denoted by several letters such as : F, P, N, … etc.



In the SI system of units load is measured in Newton's, although a single Newton, in engineering terms, is a very small load. In most engineering applications, therefore, loads appear in SI multiples, i.e. Kilo-Newton (kN) or Mega-Newton (MN).

There are a number of different ways in which load can be applied to a member. Typical loading types are:

(a) Static or dead loads, i.e. non-fluctuating loads, generally caused by gravity effects.

(b) Live loads, as produced by, for example, lorries crossing a bridge.

(c) Impact or shock loads caused by sudden blows.

(d) Fatigue, fluctuating or alternating loads, the magnitude and sign of the load changing with time.

Direct or normal stress (□)

When a bar is subjected to a uniform tension or compression, i.e. a direct force (F or P), which is uniformly or equally applied across the cross-section, then the internal forces set up are also distributed uniformly and the bar is said to be subjected to a uniform direct or normal stress, the stress being defined as

A

PP

$$\text{stress}\,(\sigma) = \frac{\text{load}}{\text{area}} = \frac{P}{A}$$

A*PP*

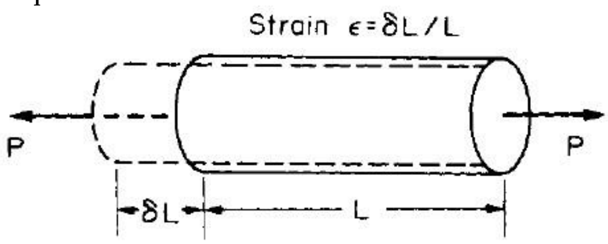
Stress (□) may thus be compressive or tensile depending on the nature of the load and will be measured in units of Newton per square metre (N/m^2) or multiples of this.

Direct strain (□)

If a bar is subjected to a direct load, and hence a stress, the bar will change in dimension (such as length). If the bar has an original length L and changes in length by an amount □L, the strain produced is defined as follows:

$$\text{strain}\,(\varepsilon) = \frac{\text{change in length}}{\text{original length}} = \frac{\delta L}{L}$$

Strain is thus a measure of the deformation of the material and is nondimensional, i.e. it has no units; it is simply a ratio of two quantities with the same unit.



Since, in practice, the extensions of materials under load are very small, it is often convenient to measure the strains in the form of strain x 10^{-6} i.e. Microstrain, when the symbol used becomes □□ . Alternatively, strain can be expressed as a percentage strain

$$\text{strain}\,(\varepsilon) = \frac{\delta L}{L} \times 100\,\%$$

Sign convention for direct stress and strain

Tensile stresses and strains are considered POSITIVE in sense producing an increase in length. Compressive stresses and strains are considered NEGATIVE in sense producing a decrease in length.

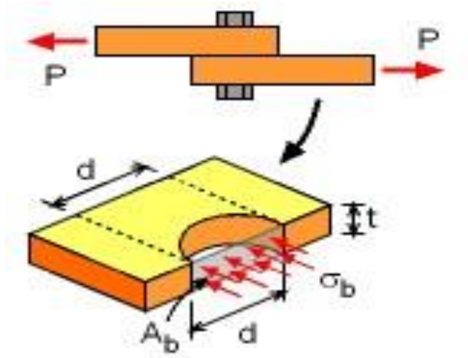
Bearing stress (□)

Bearing stress is the contact pressure between the separate bodies. It differs from compressive stress, as it is an internal stress caused by compressive forces. Bolts, pins, and rivets create stresses in the members they connect, along the bearing surface, or surface of contact.

When one body presses against another, bearing stress occurs.

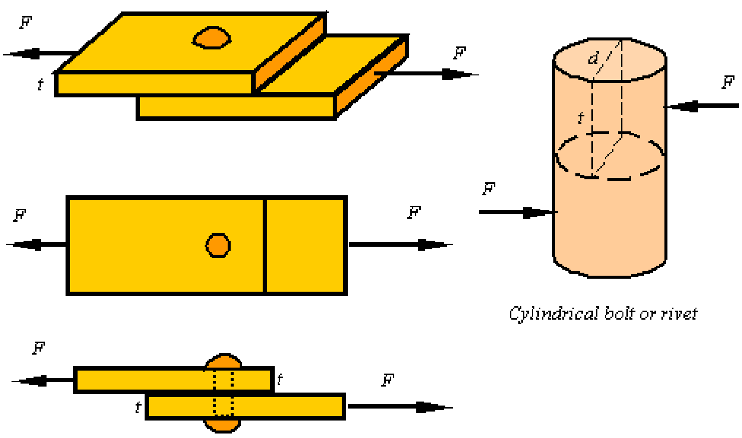
Bearing stress is given by:

$$\sigma_b = \frac{P}{A_b}$$



Bearing stress (□)

Bearing Stress – compressive force divided by the characteristic area perpendicular to it



Cylindrical bolt or rivet

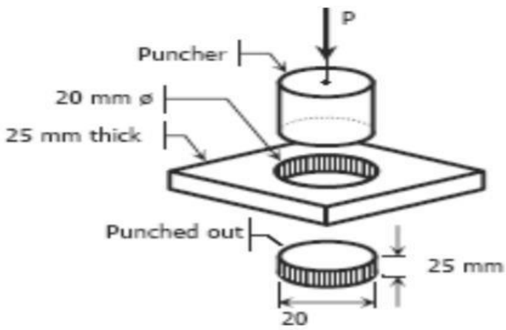
Worked example:

What force is required to punch a 20-mm-diameter hole in a plate that is 25 mm thick? The strength is 350 MN/m^2 .

Solution

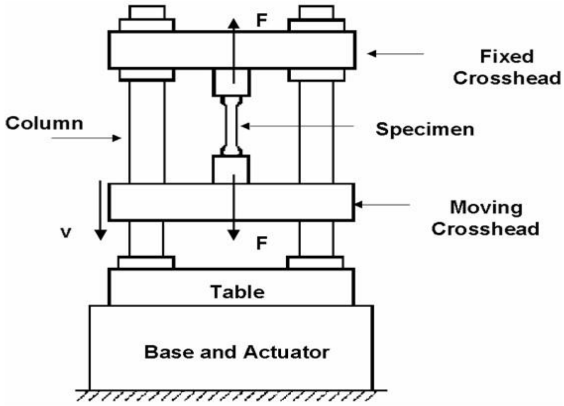
The resisting area is the shaded area along the perimeter and the shear force V is equal to the punching force P.

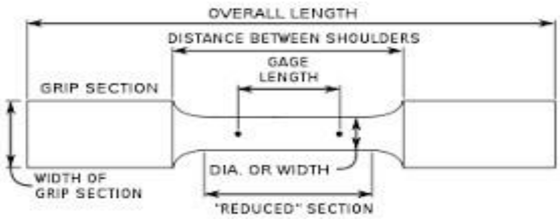
$$V = \tau A$$
$$P = 350[\pi(20)(25)]$$
$$= 549\ 778.7\ \text{N}$$
$$= 549.8\ \text{kN}$$



Tensile Test

In order to compare the strengths of various materials it is necessary to carry out some standard form of test to establish their relative properties. One such test is the standard tensile test. Measurements of the change in length of a selected gauge length of the bar are recorded throughout the loading operation by means of extensometers and a graph of load





Ductile materials

The capacity of a material to allow large extensions, referred to the ability to be drawn out plastically, is termed its ductility. Materials with high ductility are termed ductile materials, members with low ductility are termed brittle materials. A quantitative value of the ductility is obtained by measurements of the percentage elongation or percentage reduction in area, both being defined below.

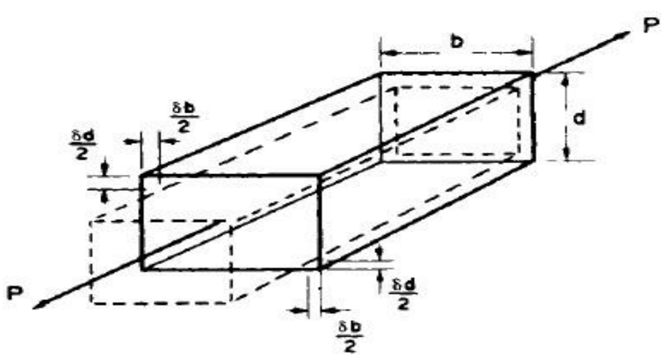
$$\text{Percentage elongation} = \frac{\text{increase in gauge length to fracture}}{\text{original gauge length}} \times 100$$

$$\text{Percentage reduction in area} = \frac{\text{reduction in cross-sectional area of necked portion}}{\text{original area}} \times 100$$

A property closely related to ductility is malleability, which defines a material's ability to be hammered out into thin sheets. A typical example of a malleable material is lead. This is used extensively in the plumbing trade where it is hammered or beaten into corners or joints to provide a weatherproof seal

Brittle materials

A brittle material is one which exhibits relatively small extensions to fracture so that the partially plastic region of the tensile test graph is much reduced. There is little or no necking at fracture for brittle materials



Consider the rectangular bar as shown below, subjected to a tensile load. Under the action of this load the bar will increase in length by an amount of □L giving a longitudinal strain in the bar of

$$\varepsilon_L = \frac{\delta L}{L}$$

Simeon Denis Poisson (1781-1840)

The bar will also exhibit, however, a reduction in dimensions laterally, i.e. its breadth and depth will both reduce. The associated lateral strains will both be equal, will be of opposite sense to the longitudinal strain, and will be given by
Provided the load on the material is retained within the elastic range the ratio of the lateral and longitudinal strains will always be constant. This ratio is termed Poisson's ratio.

$$\textbf{Poisson's ratio } (\nu) = \frac{\textbf{lateral strain}}{\textbf{longitudinal strain}} = \frac{(-\delta d/d)}{\delta L/L}$$

The negative sign of the lateral strain is normally ignored to leave Poisson's ratio simply as a ratio of strain magnitudes. It must be remembered, however, that the longitudinal strain induces a lateral strain of opposite sign, e.g. tensile longitudinal strain induces compressive lateral strain.

Poisson's ratio

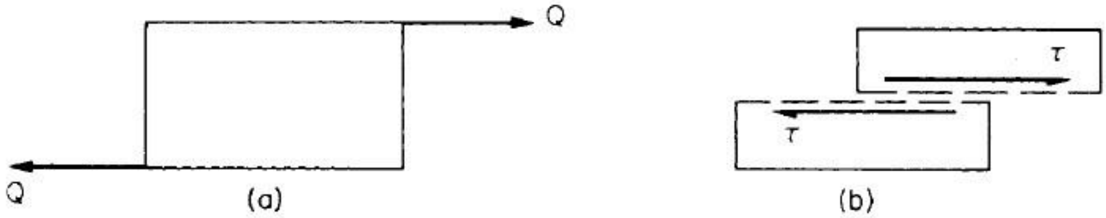
For most engineering materials the value of □ lies between 0.25 and 0.33.

stress

Consider a block or portion of material as shown below (a), subjected to a set of equal and opposite forces Q. (Such a system could be realised in a bicycle brake block when contacted with the wheel.) There is a tendency for one layer of the material to slide over another to produce the form of failure shown in (b). If this failure is restricted, then a shear stress (□) is set up, defined as follows:

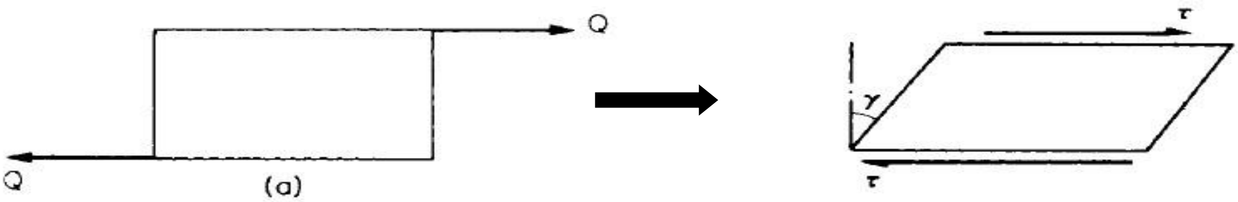
$$\text{shear stress } (\tau) = \frac{\text{shear load}}{\text{area resisting shear}} = \frac{Q}{A}$$

This shear stress will always be tangential to the area on which it acts; direct stresses, however, are always normal to the area on which they act.



strain

If one again considers the block to be a bicycle brake block it is clear that the rectangular shape of the block will not be retained as the brake is applied and the shear forces introduced. The block will in fact change shape or "strain" into the form shown below. The angle of deformation (□) is then termed the shear strain .



Shear strain is measured in radians and hence is non-dimensional, i.e. it has no units. For materials within the elastic range the shear strain is proportional to the shear stress producing it, i.e.

$$\frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \text{constant} = G$$

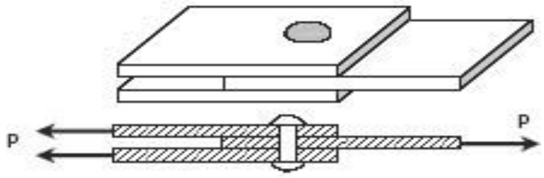
strain

The constant G is termed the modulus of rigidity or shear modulus and is directly comparable to the modulus of elasticity used in the direct stress application. The term modulus thus implies a ratio of stress to strain in each case.

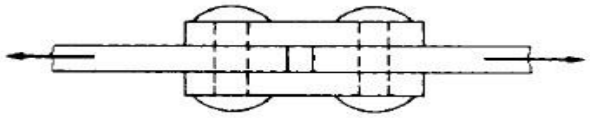
Single shear and Double shear

Consider the simple riveted lap joint. When load is applied to the plates the rivet is subjected to shear forces tending to shear it on one plane as indicated.

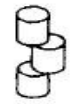
In the butt joint with two cover plates, however, each rivet is subjected to possible shearing on two faces, i.e. double shear.



Double Shear



Butt joint with two cover plates



Failure of rivet in double shear

Double shear

In such cases twice the area of metal is resisting the applied forces so that the shear stress set up is given by

$$\text{shear stress } \tau \text{ (in double shear)} = \frac{P}{2A}$$

Temperature stresses

When the temperature of a component is increased or decreased the material respectively expands or contracts. If this expansion or contraction is not resisted in any way then the processes take place free of stress. If, however, the changes in dimensions are restricted then stresses termed temperature stresses will be set up within the material.

Consider a bar of material with a linear coefficient of expansion α. Let the original length of the bar be L and let the temperature increase be t. If the bar is free to expand the change in length would be given by

$$\Delta L = L\alpha t$$

and the new length

$$L' = L + L\alpha t = L(1 + \alpha t)$$

Temperature stresses

If this extension were totally prevented, then a compressive stress would be set up equal to that produced when a bar of length L (1 + αt) is compressed through a distance of L αt. In this case the bar experiences a compressive strain

$$\varepsilon = \frac{\Delta L}{L} = \frac{L\alpha t}{L(1 + \alpha t)}$$

In most cases at is very small compared with unity so that

$$\varepsilon = \frac{L\alpha t}{L} = \alpha t$$

$$\frac{\sigma}{\varepsilon} = E$$

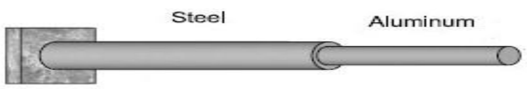
$$\text{stress } \sigma = E\varepsilon = E\alpha t$$

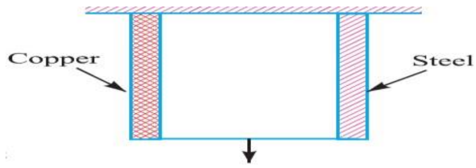
This is the stress set up owing to total restraint on expansions or contractions caused by a temperature rise, or fall, *t*. In the former case the stress is compressive, in the latter case he stress is tensile.

Chapter 2 Compound Bars

Statically Indeterminate Systems

When a system comprises two or more members of different materials, the forces in various members cannot be determined by the principle of statics alone. Such system are know as (





Statically indeterminate).

Additional equations are required to determine the unknown forces.

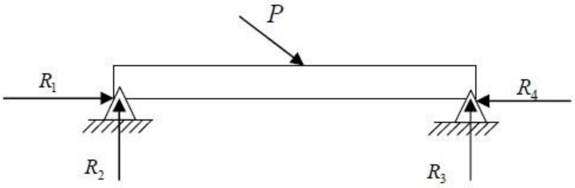
These equations are obtained from deformation condition of the system and are known as compatibility equations.

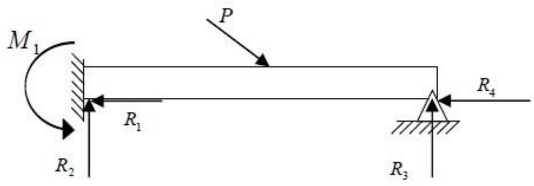
A compound bar is a case of an indeterminate system

Statically Indeterminate Systems

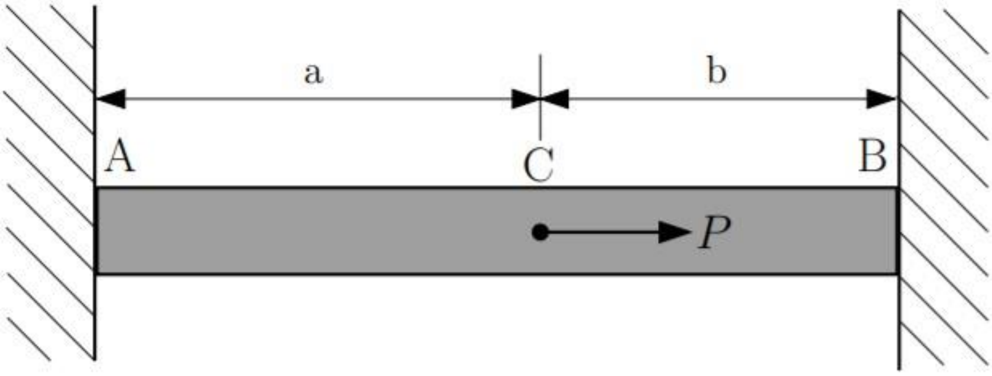
Structures where forces can be determined using the static equilibrium equations alone ($\Sigma Fx = 0$, $\Sigma Fy = 0$ and $\Sigma MA = 0$) are called statically determinate structures.

The Structures where the forces cannot be determined in this way are called statically indeterminate structures.





Example Solution



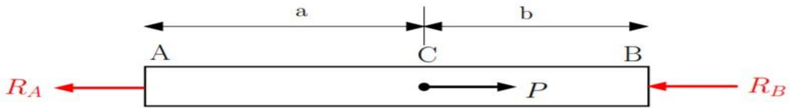
Consider a bar *AB* supported at both ends by fixed supports, with an axial force of *P* kN applied at *C* as illustrated. Find the reactions at the walls.

Draw a F.B.D

$$\delta = \frac{PL}{AE}$$

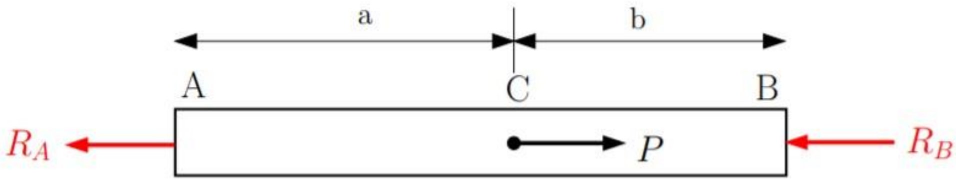
Solution

$$R_A + R_B - P = 0$$
$$\Rightarrow R_A = P - R_B$$



$$\delta_{AC} + \delta_{CB} = 0$$

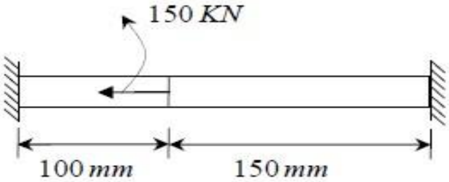
$$\frac{R_A \times a}{AE} + \frac{-R_B \times b}{AE} = 0$$



$$R_B = \left(\frac{a}{a+b}\right) P$$

$$aR_A = bR_B$$
$$a\left(P - R_B\right) = bR_B$$
$$aP = \left(a+b\right) R_B$$
$$R_A = \left(\frac{b}{a+b}\right) P$$

Example **:** A square bar ***50 mm*** on a side is held rigidly between the walls and loaded by an axial force of ***150 KN*** as shown. Determine the reactions at the end of the bar and the extension of the right portion. Take ***E=200 GPa***.



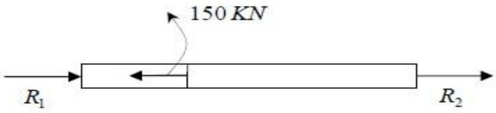
$R_1+R_2=150\times10^3$ ....(1)

$\delta_1=\delta_2$

$$\frac{R_1\times100\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}=\frac{R_2\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}$$

$0.1R_1=0.15R_2$

$R_1=1.5R_2$ ....(2)



From equations (1) and (2)

$1.5R_2+R_2=150\times10^3$ $\qquad$ $R_2=60000$ N

$R_1=90000$ N

$$\delta_2=\frac{R_2\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}=\frac{60000\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}$$

$\delta_2=0.000018$ m $\qquad$ $\delta_2=0.018$ mm

Consider a compound bar consisting of *n* members. Let all members have a common extension (*x*)

For the n^{th} member,

$$\frac{\text{stress}}{\text{strain}}=E_n=\frac{F_nL_n}{A_nx_n}$$

where F, is the force in the n^{th} member and A_n and L_n are cross-sectional area and length, respectively.

The total load (W) carried will be the sum of all such loads for all the members

$$W=\sum F_n \longrightarrow W=\sum\frac{E_nA}{L}$$

$$F_n=\frac{E_nA_nx}{L_n}$$

Compound bars subjected to external load

Now, the force in member 1 is given by

$$F_1 = \frac{E_1 A_1 x}{L_1}$$

$$x = \frac{W}{\sum \frac{E_n A_n}{L_n}} \longrightarrow F_1 = \frac{\frac{E_1 A_1}{L_1}}{\sum \frac{EA}{L}} W$$

i.e. each member ($F_1+F_2+F_3+F_4+.......+F_n$) carries a portion of the total load W proportional to its (EA / L) value.

If the bars are all of equal length the above equation reduces to

$$F_1 = \frac{E_1 A_1}{\Sigma EA} W$$

When a compound bar is constructed from members of different materials, lengths and areas and is subjected to an external tensile or compressive load W, the load (F) carried by any single member is given by

$$F_1 = \frac{\frac{E_1 A_1}{L_1}}{\sum \frac{EA}{L}} W$$

The numerator: $\frac{E_1 A_1}{L_1} \longrightarrow$ Single member

The denominator $\sum \frac{EA}{L} \longrightarrow$ Sum of all such quantiti

Equivalent or combined modulus

Consider a single compound bar of an imaginary material with an equivalent or combined modulus, *E*. Assume the extension and the original lengths of the individual members of the compound bar are the same (Equally strain).

Total load on compound bar $W = F_1 + F_2 + F_3 + F_4 + \ldots\ldots + F_n$ but
(force= stress x area)

The equivalent or combined *E* of the single *bar*, E_c , *is* .

$$\text{combined } E = \frac{E_1A_1 + E_2A_2 + \ \ldots \ + E_nA_n}{A_1 + A_2 + \ \ldots \ + A_n}$$

$$E_c = \frac{\Sigma EA}{\Sigma A}$$

Equivalent or combined modulus

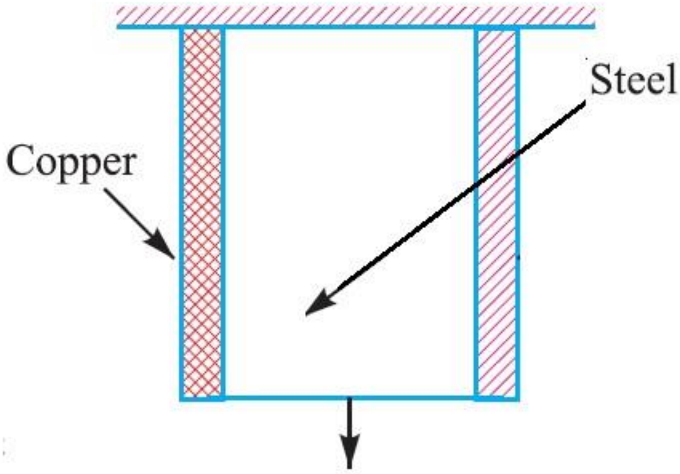
<u>With an external load *W* applied:</u>

-Stress in the equivalent bar $= \dfrac{W}{\Sigma A}$

-Strain in the equivalent bar

$$= \frac{W}{E_c \Sigma A} = \frac{x}{L}$$

Since,
In order to determine the stresses in a compound bar composed of two or more members of *different free lengths*, two principles are used:



1- The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to the long member by the short member.

2- The extension (□) of the short member plus the contraction of the long member equals this difference in free lengths

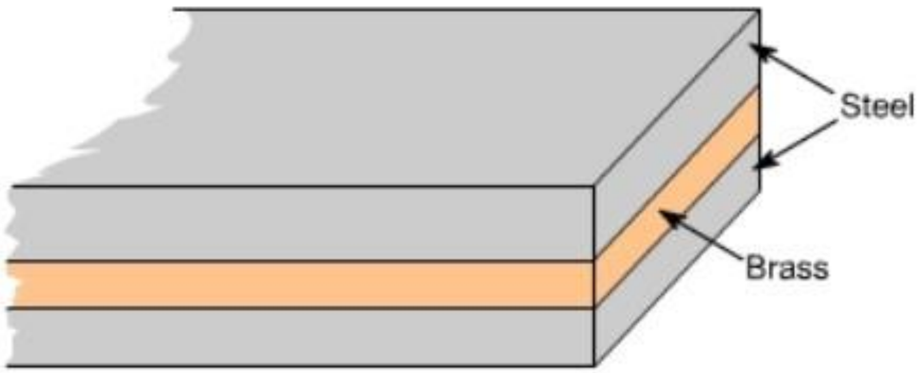
This difference in free lengths may result from:

-Applying load in tension or compression

-Temperature change in two members of different material (i.e. different coefficients of expansion) but of equal length initially.

Compound bars subjected to temperature change

Consider a compound bar constructed from two different materials (Steel & Brass) rigidly joined together



In general, the coefficients of expansion (□) of the two materials forming the compound bar will be different so that as the temperature rises each material will attempt to expand by different amounts.

Compound bars subjected to temperature change

The extension of any length L is given by $\alpha L t$

The difference of "free" expansion lengths or so-called free lengths

Compound bars subjected to temperature change

It will be seen that the following rules

holds: Rule 1:

Extension of Steel + Compression of Brass = difference in free lengths.

(Extension of short member + Compression of long member = difference in free lengths)

Rule 2:
Applying Newton law:
The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to the long member by the short member. Thus:

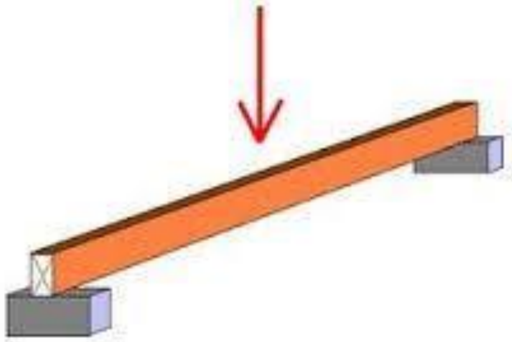
Tensile force in Steel (F_{steel}) = Compressive force in Brass (F_{brass})

Compound bars subjected to temperature change

Chapter 3 Shearing Force and Bending Moment Diagrams

What are beams

A structural member which is long when compared with its lateral dimensions, subjected to transverse forces so applied as to induce bending of the member in an axial plane, is called a beam.

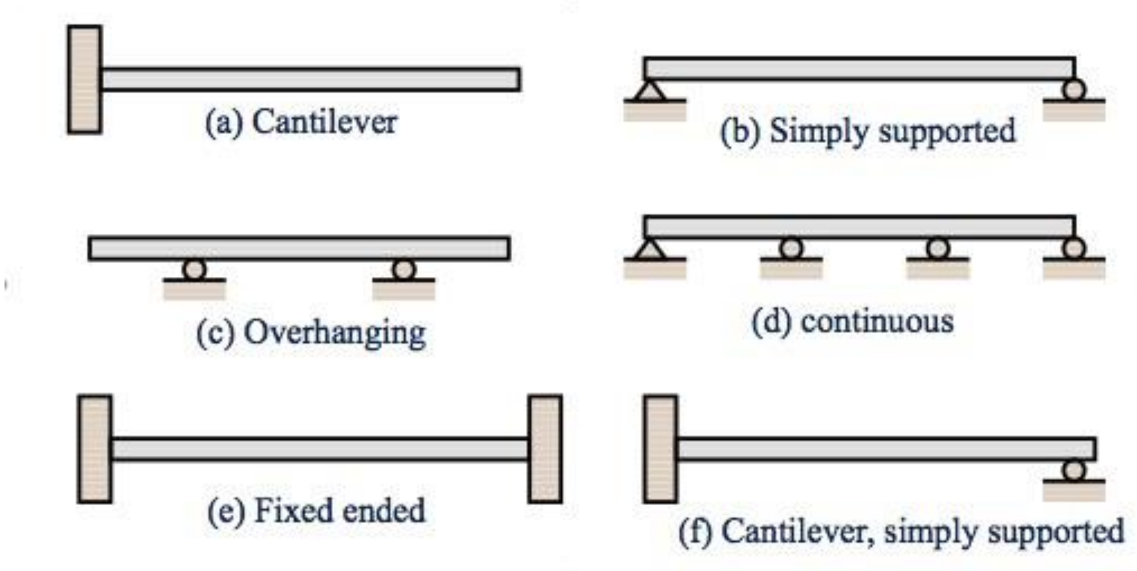


When a beam is loaded by forces or couples, stresses and strains are created throughout the interior of the beam. To determine these stresses and strains, the internal forces and internal couples that act on the cross sections of the beam must be found.

The resultant of the stresses acting on the cross section can be reduced to a shear force (S.F.) and a bending moment (B.M.) .

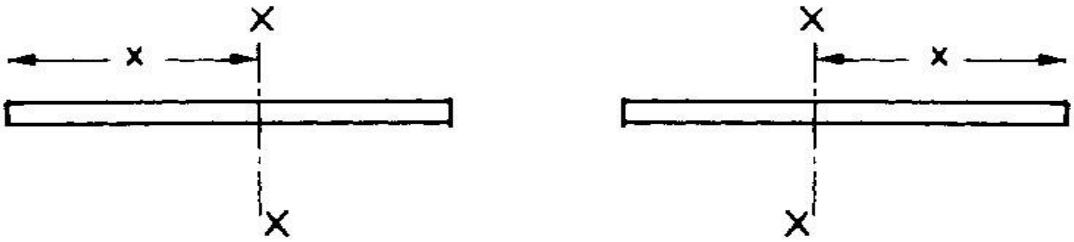
Beam Types

Types of beams- depending on how they are supported.



Shear Force (S.F.) :

The shearing force at the section is defined as the algebraic sum of the vertical forces acting (taken on one side of the section) to the left or right of the section.

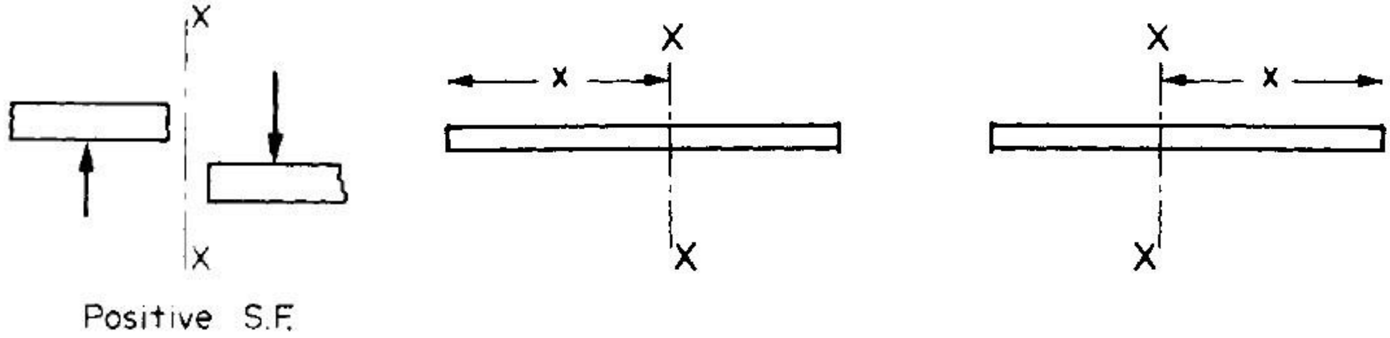


Bending Moment (B.M.):

The bending moment at the section is defined as the algebraic sum of the moment of the forces (taken on either side of the section) to the left or to the right of the section.

Shearing force (S.F.) sign convention

Forces upwards to the left of a section or downwards to the right of the section are positive.



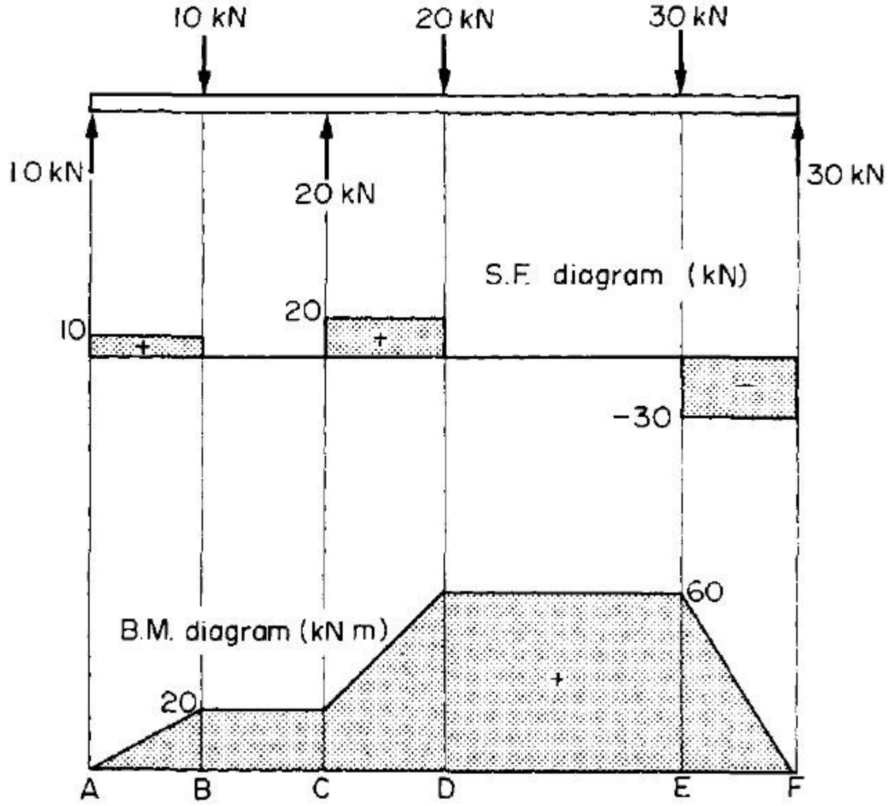
The values of the reactions at the ends of the beam may be calculated by applying both normal and vertical equilibrium conditions (Taking a moment and summation of forces).

$$R_A \times 12 = (10 \times 10) + (20 \times 6) + (30 \times 2) - (20 \times 8) = 120$$

$$R_A = 10\,\mathrm{kN}$$

For Vertical equilibrium
Summing up the forces on either side of X-X, using the sign convention listed above, the shear force at X-X is therefore +20 kN, i.e. The resultant force at X-X tending to shear the beam is 20 kN

All the above values have been calculated from the moments of the forces to the left of each section considered except for E where forces to the right of the section are taken.



Consider the simply supported beam carrying a uniformly distributed load (u.d.l), w = 25 kN/m across the complete span.

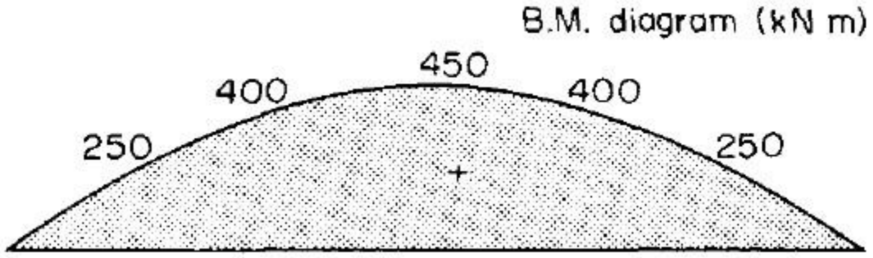
When evaluating B.M.'s it is assumed that a u.d.l. can be replaced by a concentrated load of equal value acting at the middle of its spread. When taking moments about C, therefore, the portion of the u.d.l. between A and C has an effect equivalent to that of a concentrated load of 25 x 2 = 50 kN acting the centre of AC, i.e. 1 m from C.

$$\text{B.M. at } C = (R_A \times 2) - (50 \times 1) = 300 - 50 = 250\,\text{kN m}$$

Similarly, for moments at D the u.d.l. on AD can be replaced by a concentrated load of 25 x 4 = 100 kN at the centre of AD, i.e. at C.

$$\text{B.M. at } D = (R_A \times 4) - (100 \times 2) = 600 - 200 = 400\,\text{kN m}$$

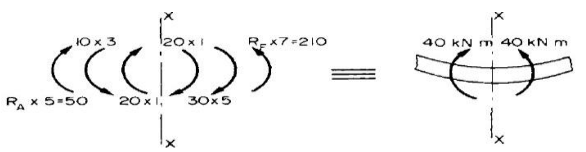
$$\text{B.M. at } E = (R_A \times 6) - (25 \times 6)3 = 900 - 450 = 450\,\text{kN m}$$



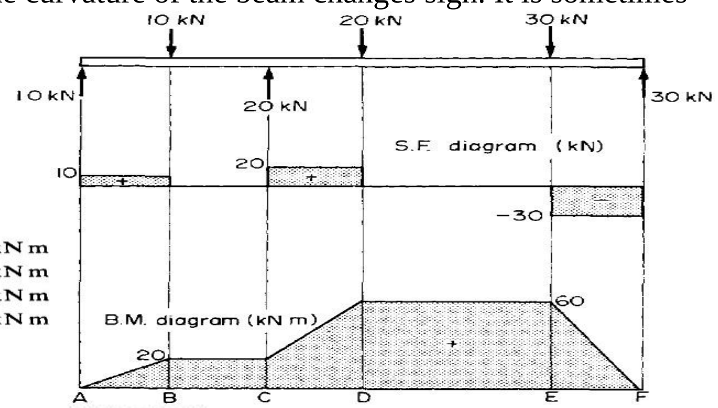
S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads (u.d.l.)

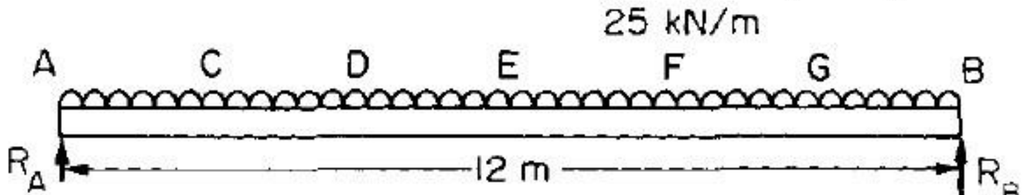
Point of Contraflexure

A point of contraflexure is a point where the curvature of the beam changes sign. It is sometimes



B.M. at A	=	0
B.M. at $B = +(10 \times 2)$	=	$+20$ kN m
B.M. at $C = +(10 \times 4) - (10 \times 2)$	=	$+20$ kN m
B.M. at $D = +(10 \times 6) + (20 \times 2) - (10 \times 4)$	=	$+60$ kN m
B.M. at $E = +(30 \times 2)$	=	$+60$ kN m
B.M. at F	=	0





referred to as a point of inflexion and will be shown later to occur at the point, or points, on the beam where the B.M. is zero.

For the beam, the B.M. diagram that this point lies somewhere between C and D (B.M. at C is positive, B.M. at D is negative). If the required point is a distance x from C then at that point.

The point of contraflexure must be situated at 1.96 m to the right of C.
Consider the beam AB shown carrying a uniform loading intensity (u.d.l) of w kN/m. By symmetry, each reaction takes half the total load, i.e., wL/2.

$$\begin{aligned}\text{B.M.} &= (42.5)(5+x) - (10\times 2)(4+x) \\ &\quad - 20(3+x) - 20x - \frac{20x^2}{2} \\ &= 212.5 + 42.5x - 80 - 20x \\ &\quad - 60 - 20x - 20x - 10x^2 \\ &= 72.5 - 17.5x - 10x^2\end{aligned}$$

Thus the B.M. is zero where

$$0 = 72.5 - 17.5x - 10x^2$$
$$x = 1.96 \text{ or } -3.7$$

Chapter 4

Bending

Simple bending theory

Consider a beam initially unstressed and subjected to a constant B.M. along its length, i.e. pure bending, as would be obtained by applying equal couples at each end, it will bend to a radius R. The top fibres of the beam will be subjected to tension and the bottom to compression. Between the two there are points at which the stress is zero. The locus of all such points is termed the Neutral axis. The radius of curvature R is then measured to this axis. For symmetrical sections the N.A. is the axis of symmetry, but whatever the section the N.A. will always pass through the centre of area or Centroid, ((the centre of area of the section through which the N.A., or axis of zero stress, is always found to pass)).

Before apply B.M

Consider now two cross-sections of a beam, HE and GF, originally parallel. When the beam is bent it is assumed that these sections remain plane; i.e. H'E' and G'F', the final positions of the sections, are still straight lines. They will then subtend some angle □.
Consider now some fibre AB in the material, distance y from the N.A. When the beam is bent this will stretch to A'B'.

$$\text{Strain in fibre } AB = \frac{\text{extension}}{\text{original length}} = \frac{A'B' - AB}{AB}$$

But $AB = CD$, and, since the N.A. is unstressed, $CD = C'D'$.

Equating the two equations for strain,

$$\frac{\sigma}{E} = \frac{y}{R} \longrightarrow \frac{\sigma}{y} = \frac{E}{R}$$

Consider now a cross-section of the beam. From equation of the stress on a fibre at distance y from the N.A. is

$$\sigma = \frac{E}{R} y$$

If the strip is of area □A the force on the strip is

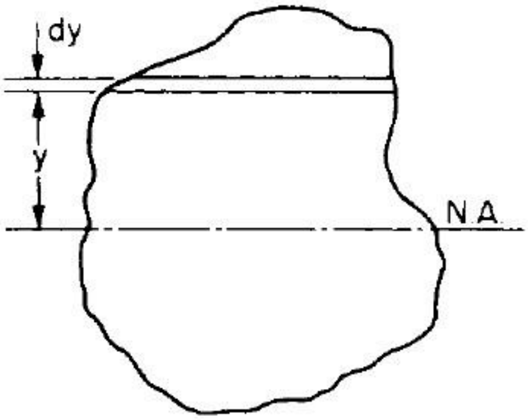
$$F = \sigma\delta A = \frac{E}{R} y\delta A$$

This has a moment about the N.A. Of

$$Fy = \frac{E}{R} y^2\delta A$$

The total moment for the whole cross-section is therefore

$$M = \sum \frac{E}{R} y^2\delta A \longrightarrow$$



Since E and R are assumed constant. The term □y^2 □A is called the second moment of area of the cross-section and given the *I*

$$M = \frac{E}{R} I \quad \text{and} \quad \frac{M}{I} = \frac{E}{R}$$

$$\Longrightarrow \frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

□The Bending theory equation

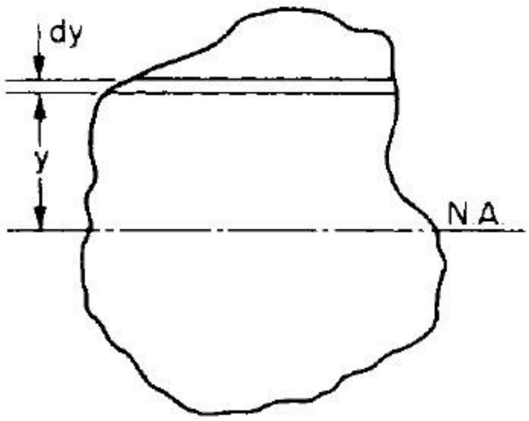
Neutral axis

In bending, one surface of the beam is subjected to tension and the opposite surface to compression there must be a region within the beam cross-section at which the stress changes sign, i.e. where the stress is zero, and this is termed the (Neutral axis).

$$\sigma = \frac{M}{I} y$$

At any section, the stress is directly proportional to y, the distance from the N.A., i.e. a varies linearly with y, the maximum stress values occurring in the outside surface of the beam where y is a maximum.

The general beam cross-section of Figure in which the N.A. Is located at some arbitrary position. The force on the small element of area is □dA acting perpendicular to the cross-section, i.e. parallel to the beam axis. The total force parallel to the beam axis is therefore □□dA.



When the beam is in equilibrium there can be no resultant force across the section, i.e. the tensile force on one side of the N.A. must exactly balance the compressive force on the other side.

This integral is the first moment of area of the beam cross-section about the N.A. since y is always measured from the N.A.

$$\int \sigma dA = 0$$

$$\int \frac{E}{R} y dA = 0 \quad \text{and hence} \quad \frac{E}{R} \int y dA = 0$$

Typical stress distributions in bending are shown below . Material near the N.A. is always subjected to relatively low stresses compared with the areas most removed from the axis.

Beams with I- or T-sections find considerable favour in present engineering applications, such as girders, where bending plays a large part.

I- Beam section T- B

Typical bending stress distribution

Section modulus

The maximum stress obtained in any cross-section is given by:

$$\sigma_{max} = \frac{M}{I} y_{max}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

Where ($Z = I / y_{max}$) termed the

$$M = \frac{I}{y_{max}} \sigma_{max} \longrightarrow M = Z \sigma_{max}$$

section modulus.

For unsymmetrical sections such as T-sections where the values of y_{max} will also be different on each side of the N.A. and here two values of section modulus are:

$$Z_1 = I/y_1 \quad \text{and} \quad Z_2 = I/y_2$$

Second moment of area

The second moment of area *I* for rectangular beam cross-section with an element of area dA, thickness dy, breadth B and distance y from the N.A. which by symmetry passes through the centroid is defined as:

Where is *I* about an axis through the lower edge of the section, with integral limits of 0 to D, defined as:

$$I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3}$$

Second moment of area

For symmetrical sections, such as I-section

Second moment of area

For unsymmetrical sections, such as T-section

$$I_{N.A.} = \underset{\text{(about } DC)}{I_{ABCD}} - \underset{\text{(about } DC)}{I_{\text{shaded areas}}} + \underset{\text{(about } HG)}{I_{EFGH}}$$

It is more convenient to divide the section into rectangles with their edges in the N.A., when the second type of standard form may be applied.

Each of these quantities may be written in the form $BD^3/3$).

It is possible to determine the second moment of area of each rectangle about an axis through its own Centroid ($I_G = BD^3/12$) to "shift" this value to the equivalent value about the N.A. by means of the "parallel axis theorem"

$$I_{N.A.} = I_G + Ah^2$$

Chapter 5 Shear Stress Distribution

Consider the case of two rectangular-sectioned beams lying one on top of the other a supported on simple supports as shown in Fig. . If some form of vertical loading is appl the beams will bend as shown in Fig. , i.e. if there is negligible friction between the mati surfaces of the beams each beam will bend independently of the other and as a result the lov surface of the top beam will slide relative to the upper surface of the lower beam.

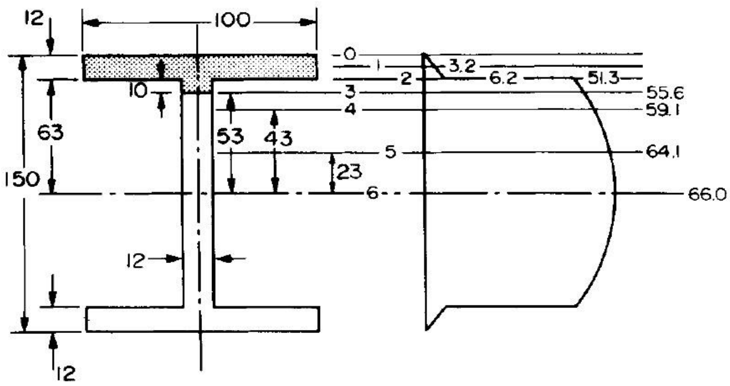
Example 1

At a given position on a beam of uniform I-section the beam is subjected to a shear force of 100 kN. Plot a curve to show the variation of shear stress across the section and hence determine the *ratio* of the maximum shear stress to the mean shear stress.

Consider the I-section shown in Fig. By symmetry, the centroid of the section is at

$$I = \frac{100 \times 150^3 \times 10^{-12}}{12} - \frac{88 \times 126^3 \times 10^{-12}}{12}$$

$$= (28.125 - 14.67)10^{-6} = 13.46 \times 10^{-6}\ \text{m}^4$$



Section	$A \times 10^{-6}$ (m²)	$\bar{y} \times 10^{-3}$ (m)	$b \times 10^{-3}$ (m)	$\tau = \frac{QA\bar{y}}{Ib}$ (MN/m²)
0	0	–	–	–
1	100 × 6 = 600	72	100	3.2
2	100 × 12 = 1200	69	100	6.2
2	1200	69	12	51.3
3	1320	68	12	55.6
4	1440	66.3	12	59.1
5	1680	61.6	12	64.1
6	1956	54.5	12	66.0

Taking moments about the top edge,

$$(100 \times 12 \times 6)10^{-9} + (10 \times 12 \times 17)10^{-9} = (100 \times 12 + 10 \times 12)h \times 10^{-9}$$

where h is the centroid of the shaded T-section,

$$7200 + 2040 = (1200 + 120)h$$

$$h = \frac{9240}{1320} = 7\,\text{mm}$$

$$\therefore \quad \bar{y}_3 = 75 - 7 = 68\,\text{mm}$$

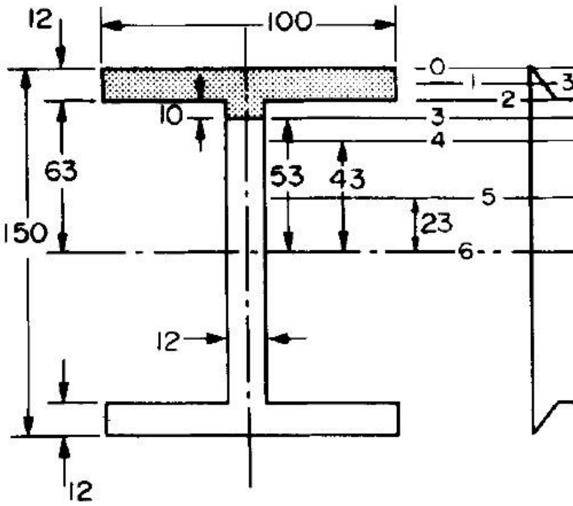
The distribution of shear stress due to bending, giving a maximum shear stress of $\tau_{max} = 66\ \text{MN/m}^2$.

Now the mean shear stress across the section is:

$$\tau_{mean} = \frac{\text{shear force}}{\text{area}} = \frac{100 \times 10^3}{3.912 \times 10^{-3}}$$

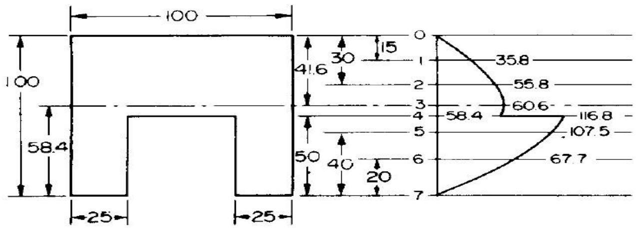
$$= 25.6\ \text{MN/m}^2$$

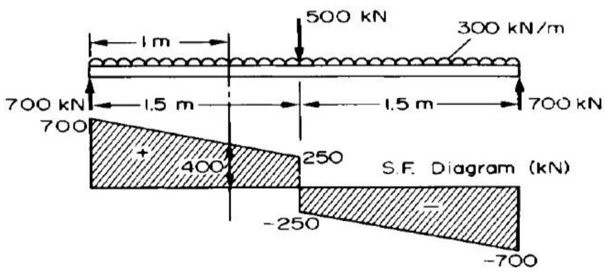
$$\frac{\text{max. shear stress}}{\text{mean shear stress}} = \frac{66}{25.6} = \mathbf{2.58}$$



Example 2

At a certain section a beam has the cross-section shown in Fig.. The beam is simply supported at its ends and carries a central concentrated load of 500 kN together with a load of 300 kN/m uniformly distributed across the complete span of 3 m. Draw the shear stress distribution diagram for a section 1 m from the left-hand support.



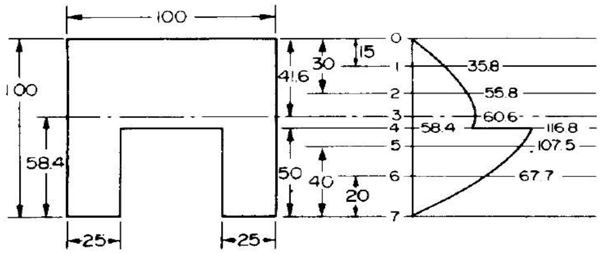


To find the position of the N.A. of the beam section take moments of area about the base.

$$(100 \times 100 \times 50)10^{-9} - (50 \times 50 \times 25)10^{-9} = (100 \times 100 - 50 \times 50)\,\bar{y} \times 10^{-9}$$

$$500000 - 62500 = (10000 - 2500)\,\bar{y}$$

$$\bar{y} = \frac{437500}{7500} = 58.4\,\text{mm}$$



$$I_{\text{N.A.}} = \left[\frac{100 \times 41.6^3}{3} + 2\left(\frac{25 \times 58.4^3}{3}\right) + \left(\frac{50 \times 8.4^3}{3}\right)\right]10^{-12}$$

$$= (2.41 + 3.3 + 0.0099)10^{-6} = 5.72 \times 10^{-6}\,\text{m}^4$$

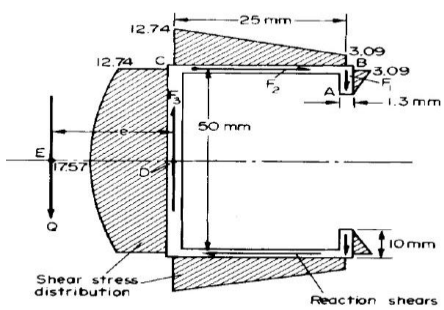
$$\tau = \frac{QA\bar{y}}{Ib} = \frac{400 \times 10^3}{5.72 \times 10^{-6}}\frac{A\bar{y}}{b} = 7 \times 10^{10}\frac{A\bar{y}}{b}$$

Section	$A \times 10^{-6}$ (m^2)	$\bar{y} \times 10^{-3}$ (m)	$b \times 10^{-3}$ (m)	$\tau = 7 \times 10^4 \frac{A\bar{y}}{b}$ (MN/m^2)
0	0	–	–	0
1	1500	34.1	100	35.8
2	3000	26.6	100	55.8
3	4160	20.8	100	60.6
4	2500	33.4	100	58.4
4	2500	33.4	50	116.8
5	2000	38.4	50	107.5
6	1000	48.4	50	67.7
7	0	–	–	0

Example 3

A beam having the cross-section shown in Fig. is constructed from materi constant thickness of 1.3 mm. Through what point must vertical loads be applie that there shall be no twisting of the section? Sketch the shear stress distributi

Let a load of QN be applied through the point E, distance e from the centre of the web.



$$I_{N.A.} = \left[\frac{1.3\times 50^3}{12} + 2\left(\frac{25\times 1.3^3}{12} + 25\times 1.3\times 25^2\right) + 2\left(\frac{1.3\times 10^3}{12} + 1.3\times 10\times 20^2\right)\right]\times 10^{-12}$$

$$= [1.354 + 2(0.00046 + 2.03) + 2(0.011 + 0.52)]10^{-8}$$

$$= 6.48\times 10^{-8}\ \mathrm{m}^4$$

Shear stress

$$\tau_A = \frac{QA\bar{y}}{Ib} = \frac{Q\times 0}{Ib} = 0$$

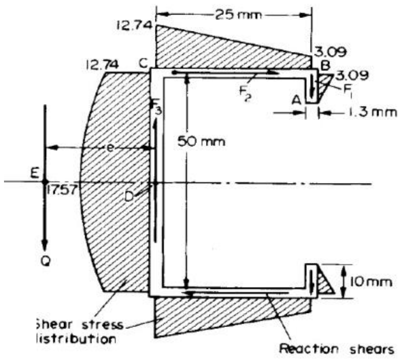
$$\tau_B = \frac{Q\times(10\times 1.3\times 20)10^{-9}}{6.48\times 10^{-8}\times 1.3\times 10^{-3}} = 3.09Q\ \mathrm{kN/m^2}$$

$$\tau_C = 3.09Q + \frac{Q(25\times 1.3\times 25)10^{-9}}{6.48\times 10^{-8}\times 1.3\times 10^{-3}}$$

$$= 3.09Q + 9.65Q = 12.74Q\ \mathrm{kN/m^2}$$

$$\tau_D = 12.74Q + \frac{Q(25\times 1.3\times 12.5)10^{-9}}{6.48\times 10^{-8}\times 1.3\times 10^{-3}}$$

$$= 12.74Q + 4.83Q = 17.57Q\ \mathrm{kN/m^2}$$



(the shear centre is the required point through which load must be applied to produce zero twist of the section).
Thus taking moments of *forces* about D for equilibrium,

$$Q\times e\times 10^{-3} = 2F_1\times 25\times 10^{-3} + 2F_2\times 25\times 10^{-3}$$

$$= 50\times 10^{-3}\left[\tfrac{1}{2}\times 3.09\,Q\times 10^3\times(10\times 1.3\times 10^{-6}) + 10^3\frac{(3.09Q + 12.74Q)}{2}(25\times 1.3\times 10^{-6})\right]$$

$$= 13.866Q\times 10^{-3}$$

$$e = 13.87\ \mathrm{mm}$$

Thus, loads must be applied through the point E, 13.87 mm to the left of the web centre-line for zero twist of the section.

When a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear (Fig.), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory to describe the behaviour of shafts subjected to torque it is necessary to make the following basic assumptions:

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Chapter 6 Torsion

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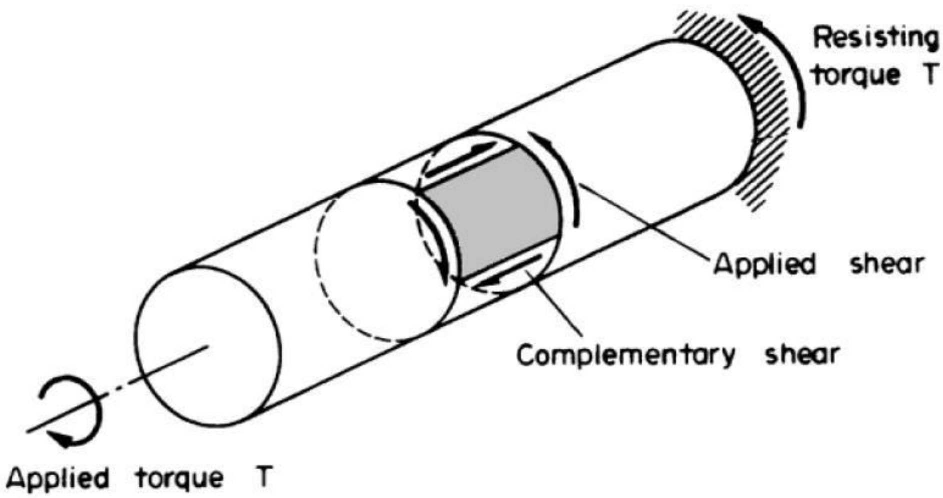
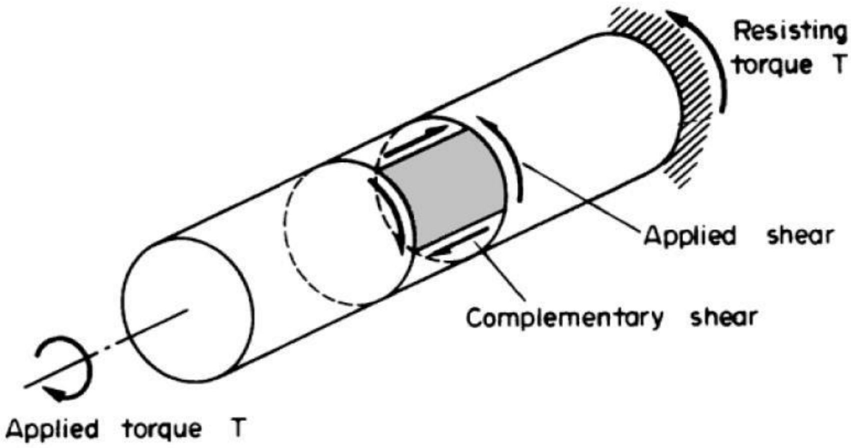


Simple Torsion Theory

The following basic assumptions:

(1) The material is homogeneous, i.e. of uniform elastic properties throughout.
(2) The material is elastic, following Hooke's law with shear stress proportional to shear strain.
(3) The stress does not exceed the elastic limit or limit of proportionality.
(4) Circular sections remain circular.
(5) Cross-sections remain plane. (This is certainly not the case with the torsion of non-circular sections.)
(6) Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.



(*a*) *Angle of twist*

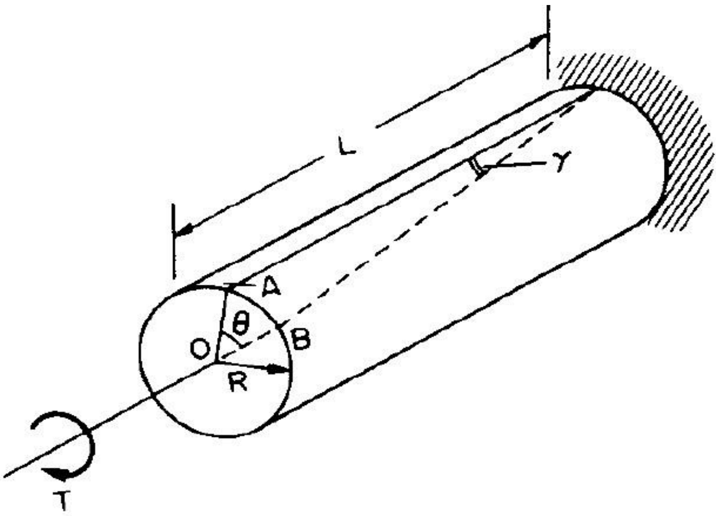
Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed (Fig.). Under the action of this torque a radial line at the free end of the shaft twists through an angle θ, point A moves to B, and AB subtends an angle γ at the fixed end. This is then the angle of distortion of the shaft, i.e. *the shear strain.*

Since

angle in radians = arc ÷ radius

$$\text{arc } AB = R\theta = L\gamma$$

$$\gamma = R\theta/L$$

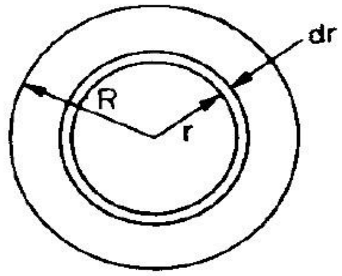


. From the definition of rigidity modulus

$$G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}$$

(*b*) *Stresses*

Let the cross-section of the shaft be considered as divided into elements of radius r and thickness dr as shown in Fig. each subjected to a shear stress τ'.



Shaft cross-section

The force set up on each element

$= \text{stress} \times \text{area}$

$= \tau' \times 2\pi r\, dr$ (approximately)

This force will produce a moment about the centre axis of the shaft, providing a contribution to the torque

$= (\tau' \times 2\pi r\, dr) \times r$

$= 2\pi\tau' r^2\, dr$

The total torque on the section T will then be the sum of all such contributions across the section,

i.e.

$$T = \int_0^R 2\pi\tau' r^2\, dr$$

Now the shear stress τ' will vary with the radius r and must therefore be replaced in terms of r before the integral is evaluated From eqn.

$$\tau' = \frac{G\theta}{L} r$$

$$T = \int_0^R 2\pi \frac{G\theta}{L} r^3\, dr \quad = \frac{G\theta}{L} \int_0^R 2\pi r^3\, dr$$

Polar second moment of area

the polar second moment of area J is defined as $J = \int_0^R 2\pi r^3\, dr$

For a solid shaft,

$$J = 2\pi\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{2\pi R^4}{4} \quad \text{or} \quad \frac{\pi D^4}{32}$$

For a hollow shaft of internal radius r,

$$J = 2\pi\int_r^R r^3\, dr = 2\pi\left[\frac{r^4}{4}\right]_r^R$$

$$= \frac{\pi}{2}(R^4 - r^4) \quad \text{or} \quad \frac{\pi}{32}(D^4 - d^4)$$

Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in Fig, their values being given by the simple torsion theory as

$$\tau = \frac{G\theta}{L}R$$

Now from the definition of the shear or rigidity modulus G,

$$\tau = G\gamma$$

Section modulus

It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\frac{T}{J}=\frac{\tau}{R} \quad \text{or} \quad \tau=\frac{TR}{J}$$

R the outside radius of the shaft

$$\tau_{max}=\frac{TR}{J} \quad \text{or} \quad \tau_{max}=\frac{T}{Z}$$

where $Z = J/R$ is termed the *polar section modulus.*

for solid shafts, $$Z=\frac{\pi D^3}{16}$$

and for hollow shafts, $$Z=\frac{\pi(D^4-d^4)}{16D}$$

Torsional rigidity

The angle of twist per unit length of shafts is given by the torsion theory as

$$\frac{\theta}{L}=\frac{T}{GJ}$$

The quantity *GJ* is termed the *torsional rigidity* of the shaft and is thus given by

$$GJ=\frac{T}{\theta/L}$$

i.e. the torsional rigidity is the torque divided by the angle of twist
(in radians) per unit length.

Composite shafts – series connection

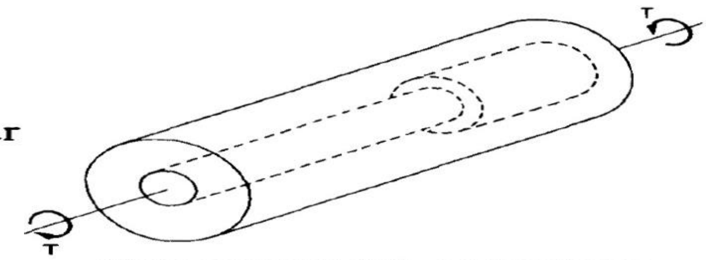
If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is termed *series-connected*

the torques in each shaft, e.g. for two shafts in series

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2}$$

the angles of twist in each shaft are equal, i.e. $\theta_1 = \theta_2$, so that for similar materials in each shaft

$$\frac{J_1}{L_1} = \frac{J_2}{L_2} \quad \text{or} \quad \frac{L_1}{L_2} = \frac{J_1}{J_2}$$



"Series-connected" shaft – common torque.

Composite shafts – parallel connection

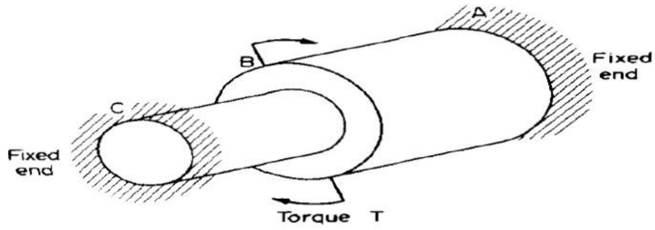
If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be *connected in parallel*.

For parallel connection,
total torque $T = T_1 + T_2$
In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2}$$

for equal lengths

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2}$$



"Parallel-connected" shaft – shared torque.

Power transmitted by shafts

If a shaft carries a torque T Newton metres and rotates at ω rad/s it will do work at the rate of

$T\omega$ Nm/s (or joule/s)

the basic unit of power being the **Watt** (1 **Watt** = 1 **Nm/s**).

Thus, the power transmitted by the shaft:

$= T\omega$ Watts.

Example 1

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per metre of the shaft length if $G = 80$ GN/m^2.

(b) If the shaft were now bored in order to reduce weight to produce a tube of 100 mm outside diameter and 60 mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

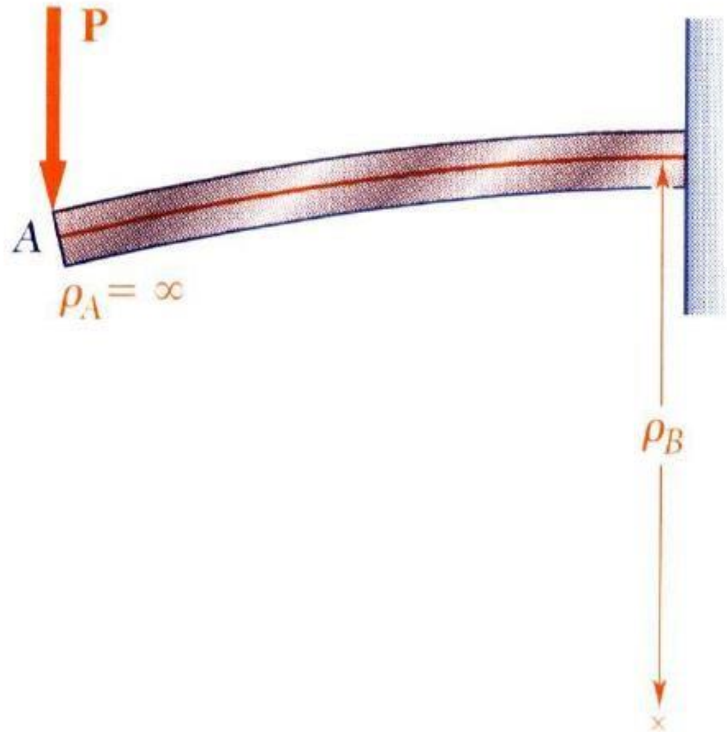
Solution

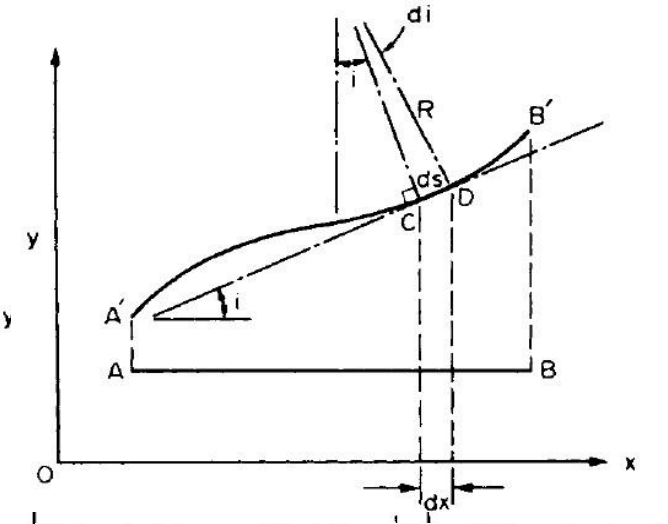
(a)

$$\text{Power} = T\omega \quad \therefore \quad \text{torque } T = \frac{\text{power}}{\omega}$$

$$T = \frac{75 \times 10^3}{150 \times 2\pi/60} = 4.77 \text{ kN m}$$

Chapter 7 Slope and Deflection





O Unloaded beam *AB* deflected to *A′B′* under load.

Unloaded beam *AB* deflected to *A′B′* under load.

$$i = \frac{dy}{dx}$$

lationship between loading, S.F., B.M., slope and deflection

Consider a beam AB which is initially horizontal when unloaded. If this deflects to a new position A′B′ under load, the slope at any point C is:

This is usually very small in practice, and for small curvatures.

$$ds = dx = Rdi$$

$$\frac{di}{dx} = \frac{1}{R}$$

Relationship between loading, S.F., B.M., slope and deflection

Now from the simple bending theory

$$i = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{R}$$

$$\frac{M}{I} = \frac{E}{R} \longrightarrow \frac{1}{R} = \frac{M}{EI}$$

And knowing that,

$$\frac{d^2y}{dx^2} = \frac{1}{R}$$

Therefore substituting

$$M = EI\frac{d^2y}{dx^2} \longrightarrow$$

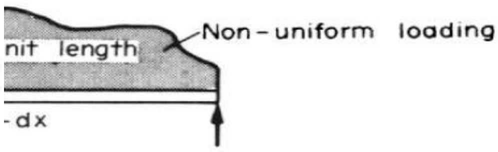
Relationship between loading, S.F., B.M., slope and deflection

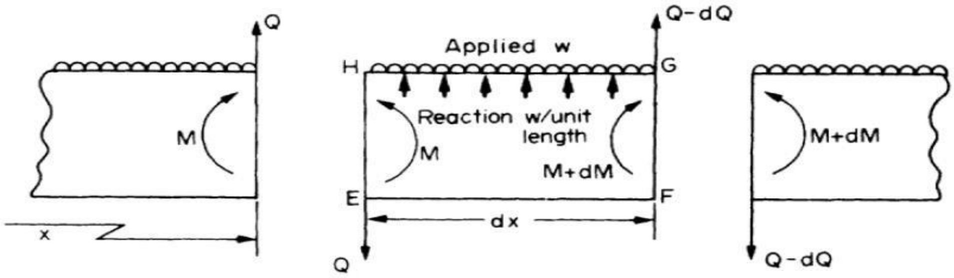
Also, for equilibrium, moments about any point must be zero. Therefore taking moments about F,

$$(M + dM) + wdx\frac{dx}{2} = M + Qdx$$

Therefore neglecting the square

$$dM = Qdx \longrightarrow M = \int Qdx$$





Relationship between

loading, S.F., B.M., slope and deflection The results can then be summarised as follows:

If the value of the B.M. at any point on a beam is known in terms of x, the distance along the beam, and provided that the equation applies along the complete beam, then integration of eqation (mension below) will results to the yield slopes and deflections at any point.

$$M = EI\frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dy}{dx} = \int \frac{M}{EI}dx + A$$

$$y = \int\int\left(\frac{M}{EI}dx\right)dx + Ax + B$$

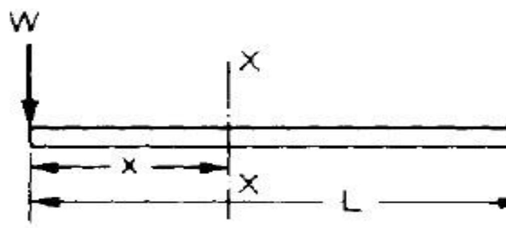
where A and B are constants of integration evaluated from known conditions of slope and deflection for particular values of x.

Direct integration method

(a) Cantilever with concentrated load at the end

$$M_{xx} = EI\frac{d^2y}{dx^2} = -Wx$$

$$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + A$$



$$EIy = -\frac{Wx^3}{6} + Ax + B$$

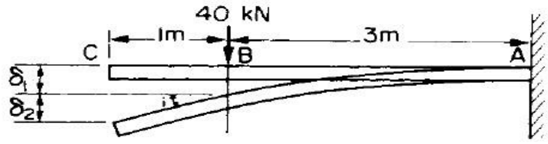
$$x = L, \quad \frac{dy}{dx} = 0 \qquad \therefore \quad A = \frac{WL^2}{2}$$

$$x = L, \; y = 0 \qquad \therefore \quad B = \frac{WL^3}{6} - \frac{WL^2}{2}L = -\frac{WL^3}{3}$$

Example 1

(a) A uniform cantilever is 4 m long and carries a concentrated load of 40 kN at a point 3 m from the support. Determine the vertical deflection of the free end of the cantilever if $EI = 65\ \text{MN m}^2$.

(b) How would this value change if the same total load were applied but uniformly distributed over the portion of the cantilever 3 m from the support?



Solution

(a) With the load in the position shown in Fig. the cantilever is effectively only 3 m long, the remaining 1 m being unloaded and therefore not bending. Thus, the standard equations for slope and deflections apply between points A and B only.

$$\text{Vertical deflection of } B = -\frac{WL^3}{3EI} = -\frac{40\times10^3\times3^3}{3\times65\times10^6} = -5.538\times10^{-3}\ \text{m} = \delta_1$$

$$\text{Slope at } B = \frac{WL^2}{2EI} = \frac{40\times10^3\times3^2}{2\times65\times10^6} = 2.769\times10^{-3}\ \text{rad} = i$$

Now BC remains straight since it is not subject to bending.

$$\delta_2 = -iL = -2.769\times10^{-3}\times1 = -2.769\times10^{-3}\ \text{m}$$

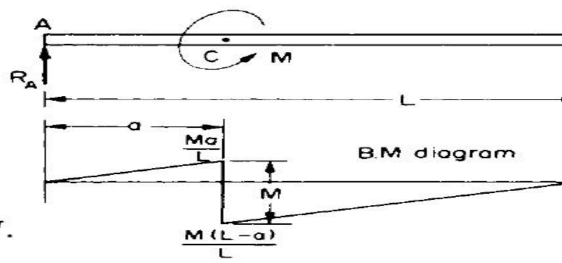
$$\text{vertical deflection of } C = \delta_1+\delta_2 = -(5.538+2.769)10^{-3} = \mathbf{-8.31\ mm}$$

The negative sign indicates a deflection in the negative y direction, i.e. downwards.

Consider the beam AB shown with a moment or couple M applied at some point C.

$R_A = \dfrac{M}{L}$ upwards,

$R_B = \dfrac{M}{L}$ downwards



For sections between A and C the B.M. is $\dfrac{M}{L}x$.

For sections between C and B the B.M. is $\dfrac{Mx}{L} - M$.

(b) How would this value change if the same total load were applied but uniformly distributed over the portion of the cantilever 3 m from the support?

(b) With the load uniformly distributed,

$$w = \frac{40 \times 10^3}{3} = 13.33 \times 10^3 \text{ N/m}$$

Again using standard equations listed in the summary

$$\delta_1' = -\frac{wL^4}{8EI} = \frac{13.33 \times 10^3 \times 3^4}{8 \times 65 \times 10^6} = -2.076 \times 10^{-3} \text{ m}$$

$$\text{and slope } i = \frac{wL^3}{6EI} = \frac{13.33 \times 10^3 \times 3^3}{6 \times 65 \times 10^6} = 0.923 \times 10^3 \text{ rad}$$

$$\delta_2' = -0.923 \times 10^{-3} \times 1 = 0.923 \times 10^{-3} \text{ m}$$

$$\therefore \quad \text{vertical deflection of } C = \delta_1' + \delta_2' = -(2.076 + 0.923)10^{-3} = \mathbf{-3\,mm}$$

where A and B are two constants of integration.

Now when $x = 0, \quad y = 0 \quad \therefore \quad B = 0$

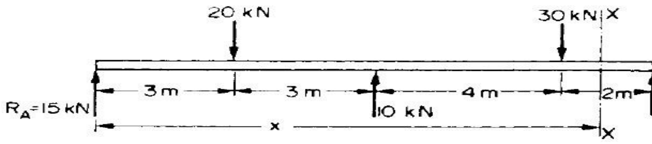
and when $x = 12, \quad y = 0$

$$0 = \frac{15 \times 12^3}{6} - 20\left[\frac{9^3}{6}\right] + 10\left[\frac{6^3}{6}\right] - 30\left[\frac{2^3}{6}\right] + 12A$$

$$= 4320 - 2430 + 360 - 40 + 12A$$

$$12A = -4680 + 2470 = -2210$$

$$A = -184.2$$



The deflection at any point is given by

$$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] - 184.2x$$