1.3 Simpson's One-Third Method

The derivative of simple Simpson's One-Third rule formula to approximate value of the integral in eq.(1) we approximate the function f(x) as Newton forward interpolation formula and then integration over the interval $[x_0,x_2]$, where

$x=x_m=x_0+mh$, $m=0,1,2,\ldots, n$

Now, let **n=2**, we have

$$\int_{a=x_0}^{b=x_2} f(x)dx = \int_0^2 f(x_m)dx\frac{dm}{dm} = \int_0^2 f(x_m)dm.h = h\int_0^2 f(x_m)dm$$

$$=h\int_0^2[f_0 + m\Delta f_0 + \frac{m(m-1)}{2!}\Delta^2 f_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 f_0 + \cdots]dm$$

$$= h\left[mf_0 + \frac{m^2}{2}\Delta f_0 + (\frac{m^3}{6}-\frac{m^2}{4})\Delta^2 f_0 + (\frac{m^4}{24}-\frac{m^3}{6}+\frac{m^2}{6})\ \ \Delta^3 f_0 + \cdots\right]_0^2$$

$$= h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 + \cdots - 0\right]$$

$$\int_{a=x_0}^{b=x_2} f(x)dx \cong \frac{h}{3}[f_0 + 4f_1 + f_2] \qquad (6)$$

Equation (6) is called **Simple Simpson's One-Third Rule formula** ,the local truncation error in equation (6) is

$$\text{L.T.E.} = \frac{-h^5}{90} f^4(\theta) \ \ , \ \ \theta\epsilon(x_0, x_2)$$

In general, we have subintervals $[x_{i-2}, x_i]$ and we apply eq.(6) at each subinterval we get:

$$\int_{x_i}^{x_{i+2}} f(x)dx \cong \frac{h}{3}[f_i + 4f_{i+1} + f_{i+2}]$$

then

$$\int_{a=x_0}^{b=x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \cdots + \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n]$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 4f_{n-1} + f_n\right]$$

$$= \frac{h}{3}[f_0 + 4\textstyle\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + f_n] \qquad (7)$$

Equation (7) is called **Composite Simpson's One-Third Rule formula** ,and the total truncation error in equation (7) is

$$E_T = \left[\frac{-h^5}{90} f^4(\theta_1)\right] + \left[\frac{-h^5}{90} f^4(\theta_2)\right] + \cdots + \left[-\frac{h^5}{90} f^4(\theta_n)\right]$$

$$E_T = -n\left[\frac{-h^5}{180} f^4(\theta\ \ )\right] = -\frac{(b-a)^5}{180n^4} f^4(\theta) \quad , \quad \theta\epsilon(a,b)$$

This is **Composite error term** in Composite Simpson's One-Third Rule.

Example(4): Evaluate the following integral using Simpson's One-Third Rule taking h=0.25.

$$\int_{0}^{1} \frac{1}{1+x^2}\,dx$$

Solution:

a=0 , b=1 , $f(x)=\frac{1}{1+x^2}$ and h=0.25 , $n=\frac{b-a}{h}=4$

$x_i=x_{i-1}+h \Longrightarrow$

$x_0=0$, $f(x_0)=f(0)=1$

$x_1=0+0.25=0.25$, $f(x_1)=f(0.25)=0.9412$

$x_2=0.25+0.25=0.5$, $f(x_2)=f(0.5)=0.8$

$x_3=0.5+0.25=0.75$, $f(x_3)=f(0.75)=0.64$

$x_4=0.75+0.25=1$, $f(x_4)=f(1)=0.5$

$$I[f]=\int_{0}^{1}\frac{1}{1+x^2}\,dx=\frac{h}{3}[f_0+4\sum_{i=1}^{n}f(x_{2i-1})+2\sum_{i=1}^{n-1}f(x_{2i})+f_n]$$

$$=\frac{0.25}{3}[1+4(0.9412)+2(0.8)+4(0.64)+0.5]=0.7854$$

$$=\frac{0.25}{2}\;[1+2(\,0.9412+0.8+0.64)+0.5]$$

$$=0.7828$$