1.6 Romberg Method

The Romberg method generally generates a lower trigonometric matrix of numbers, all of which are approximations of the value of integration

R(1,1)					
R(2,1)	R(2,2)				
R(3,1)	R(3,2)	R(3,3)			
R(4,1)	R(4,2)	R(4,3)	R(4,4)		
.	.	.	.		
R(n,1)	R(n,2)	R(n,3)	R(n,4)	…	R(n,n)

The numbers in the first column of the matrix represent the approximate initial values of integration after applying the trapezoidal rule i.e.:

$$R(1,1) = \frac{b-a}{2}[f(a) + f(b)]$$

The rest of the elements of the first column are calculated from the relationship:

$$R(k,1) = \frac{1}{2}\left[R(k-1,1) + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (i - \tfrac{1}{2})h_{k-1})\right] \;,\quad \text{k} = 2,3,\dots,\text{n}$$

$$h_{k-1} = \frac{b-a}{2^{k-2}}$$

The rest of the elements are calculated from the law:

$$R(k,i) = \frac{4^{i-1} * R(k,i-1) - R(k-1,i-1)}{4^{i-1} - 1} \quad , \quad k,i = 2,3,\dots,n$$

Example(8): Find the approximation value of the integral by using Romberg integration method

$$\int_0^2 x^2 dx \qquad , \qquad n=4$$

Solution: Clearly, a=0 , b=2 , $f(x) = x^2$

$$R(1,1) = \frac{b-a}{2}[f(a) + f(b)] = \frac{2-0}{2}[4 + 0] = 4$$

$$R(k,1) = \frac{1}{2}\left[R(k-1,1) + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (i - \frac{1}{2})h_{k-1})\right] , \quad k = 2,3,4$$

$$h_{k-1} = \frac{b-a}{2^{k-2}}$$

Let k=2 then $h_1 = \frac{2-0}{2^{2-2}} = 2$

$$R(2,1) = \frac{1}{2}\left[R(1,1) + h_1\sum_{i=1}^{2^{2-2}} f(0 + (i - \frac{1}{2})h_1)\right]$$

$$= \frac{1}{2}\left[4 + 2 * f(0 + (0 + \frac{1}{2} * 2)\right] = \frac{6}{2} = 3$$

k=3 then $h_2 = \frac{2-0}{2^{3-2}} = 1$

$$R(3,1) = \frac{1}{2}\left[R(2,1) + h_2\sum_{i=1}^{2^{3-2}} f(0 + (i - \frac{1}{2})h_2)\right]$$

$$= \frac{1}{2}\left[3 + 1 * \{f\left(0 + \frac{1}{2} * 1\right) + f\left(0 + \frac{3}{2} * 1\right)\}\right]$$

$$= \frac{1}{2}\left[3 + \frac{1}{4} + \frac{9}{4}\right] = \frac{11}{4} = 2.75$$

k=4 then $h_3 = \frac{2-0}{2^{4-2}} = \frac{1}{2}$

$$R(4,1) = \frac{1}{2}\left[R(3,1) + h_3 \sum_{i=1}^{2^{4-2}} f(0 + (i - \frac{1}{2})h_3)\right]$$

$$= \frac{1}{2}\left[2.75 + \frac{1}{2} * \{f\left(0 + \frac{1}{2} * \frac{1}{2}\right) + f\left(0 + \frac{3}{2} * \frac{1}{2}\right) + f\left(0 + \frac{5}{2} * \frac{1}{2}\right) + f\left(0 + \frac{7}{2} * \frac{1}{2}\right)\}\right]$$

$$= 2.6873$$

Now, we evaluate the values

$$R(k,i) = \frac{4^{i-1} * R(k,i-1) - R(k-1,i-1)}{4^{i-1} - 1} \quad , \quad k,i = 2,3,4$$

$$R(2,2) = \frac{4^{2-1} * R(2,1) - R(1,1)}{4^{2-1} - 1} = \frac{4 * R(2,1) - R(1,1)}{4^{2-1} - 1}$$

$$= \frac{4*3-4}{4-1} = \frac{8}{3} = 2.6667$$

$$R(3,2) = \frac{4^{2-1} * R(3,1) - R(2,1)}{4^{2-1} - 1} = \frac{8}{3} = 2.6667$$

And so on.

Example(9): Find the approximation value of the integral by using Romberg integration method

$$\int_0^{\frac{\pi}{4}} \sec x \, dx \qquad , \qquad n=5$$

Solution: H.W.

Romberg Integration Algorithm

Input: a , b , n , f(x)

Step(1): evaluate h=b-a

Step(2): find $R(1,1) = \frac{b-a}{2}[f(a) + f(b)]$

Step(3): for k=2,3,4,…,n find $h_{k-1} = \frac{b-a}{2^{k-2}}$ and

$$R(k,1) = \frac{1}{2}\left[R(k-1,1) + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (i - \tfrac{1}{2})h_{k-1})\right]$$

Step(4): for i=2,3,…,n and k=i,i+1,…,n evaluate

$$R(k,i) = \frac{4^{i-1} * R(k,i-1) - R(k-1,i-1)}{4^{i-1} - 1}$$

Step(5): print R(n,n) and stop.

Note: To increase the efficiency of the Romberg method, the approximate values are calculated as follows:

$$\begin{array}{llll}
R(1,1) & & & \\
\downarrow & & & \\
R(2,1) \rightarrow & R(2,2) & & \\
\downarrow & & & \\
R(3,1) \rightarrow & R(3,2) \rightarrow & R(3,3) & \\
\downarrow & & & \\
R(4,1) \rightarrow & R(4,2) \rightarrow & R(4,3) \rightarrow & R(4,4)
\end{array}$$

$|R(i,i) - R(i,i-1)| < \epsilon$