1.7 Gauss-Legendre Integration

If f(x) is continuous on the interval [-1,1] then Gauss-Legendre two-point rule is

$$\int_{-1}^{1} f(x)\,dx \cong G_2(f) = f(-\frac{1}{\sqrt{3}}) + f\left(\frac{1}{\sqrt{3}}\right) + E_2(f)$$

Where $E_2(f) = \frac{f^{(4)}(c)}{135}$

And Gauss-Legendre three-point rule is

$$\int_{-1}^{1} f(x)\,dx \cong G_3(f) = \frac{f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + f\left(\sqrt{\frac{3}{5}}\right)}{9} + E_3(f)$$

Where $E_3(f) = \frac{f^{(6)}(c)}{15,750}$

ملاحظة: اذا كان المطلوب ايجاد تكامل الدالة f على فترة *[a,b]* لاتساوي *[-1,1]* نستخدم change of variable

$$x = \frac{(b-a)t+(b+a)}{2}$$

$$dx = \frac{b-a}{2}dt$$

$$\int_{a}^{b} f(x)\,dx = \int_{-1}^{1} f\left(\frac{(b-a)t+(b+a)}{2}\right)(\frac{b-a}{2})dt$$

$$= \frac{b-a}{2}\sum_{k=1}^{n} w_{n,k} f\left(\frac{(b-a)t_{n,k}+(b+a)}{2}\right)$$

Example(10): Use the two-point Gauss-Legendre rule to approximate the following integral

$$\int_{-1}^{1} \frac{dx}{x+2}$$

Solution:

$$\int_{-1}^{1} \frac{dx}{x+2} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = f(-0.57735) + f(0.57735)$$

=0.70291+0.38800=1.09091

Where the exact solution is

$$\int_{-1}^{1} \frac{dx}{x+2} \ln(3) - \ln(1) \cong 1.09861$$

Example(11): Use the two-point Gauss-Legendre rule to approximate the following integral

$$\int_{0}^{2} e^{x^2} dx$$

Solution: To solve this integral we have change of variable then apply method

a=0, b=2

$$x = \frac{(b-a)t+(b+a)}{2} = \frac{(2-0)t+(2+0)}{2} = t+1$$

$$dx = \frac{b-a}{2}dt = \frac{2-0}{2}dt = dt$$

$$\int_{0}^{2} e^{x^2} dx = \frac{b-a}{2} \int_{-1}^{1} e^{(t+1)^2} dt = \int_{-1}^{1} e^{(t+1)^2} dt$$

$$\int_{0}^{2} e^{x^2} dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = e^{\left(-\frac{1}{\sqrt{3}}+1\right)^2} + e^{\left(\frac{1}{\sqrt{3}}+1\right)^2}$$

$$= 1.19558 + 12.0376 = 13.2332$$

Example(12): Use the three-point Gauss-Legendre rule to approximate the following integral $\int_0^2 e^{x^2} dx$, then use the Trapezoidal rule when n=1 and compare between the results.

Solution: H.W.
