3- Taylor Series Method

The problem to be solved is a first order ODE

$$\frac{dy(x)}{dx} = f(x, y), \quad y(x_0) = y_0$$

We use Taylor series expansions to the exact solution

$$y(x) = y(x_0) + (x - x_0)\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{\substack{x=x_0,\\ y=y_0}} + \frac{(x-x_0)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{\substack{x=x_0,\\ y=y_0}} + \ldots$$

where

$$y' = f(x, y)$$

$$y'' = f'(x, y) = f_x + f_y y' = f_x + f.f_y$$

$$y''' = f''(x, y) = f_{xx} + 2f.f_{xy} + f.(f_y)^2 + f^2.f_{yy} + f_x.f_y$$

نلاحظ ان صيغة تيلر تحتاج الى حساب مشتقات الدالة f(x,y) ومن رتب عالية وذلك للحصول على دقة جيدة . ان الصيغة العامة لمتسلسلة تيلر من الرتبة (k) (k^{th} order Taylor series formula) هي:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \ldots + \frac{h^k}{k!}y_i^{(k)} + O(h^{k+1}) \qquad (5)$$

Where $h = x - x_0$, $x_{i+1} = x_i + h$ And the local truncation error is

$$E_T = O(h^{k+1}) = \frac{h^{k+1}}{(k+1)!}y_i^{(k+1)}(\theta) \quad , \quad \theta \in (x_i, x_{i+1})$$

The Taylor series method of order k^{th} has the property that local truncation error (L.T.E.) is of order $O(h^{k+1})$,k can be chosen as large as necessary to make this error as small as desired

Example (5): Use Taylor series method of order(4) to solve approximately the initial value problem

$$y' = x - y, \quad y(0) = 1$$

At the values x=0.1 , 0.2 , 0.3 , 0.4

Solution:

$f(x,y)=x-y$, $x_0=0$, $y_0=1$, $h=0.1$

$x_{i+1} = x_i + h$

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i^{(3)} + \frac{h^4}{4!}y_i^{(4)} \qquad \text{, for} \quad i= 0, 1, 2,3$$

i=0 $\longrightarrow$ $y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0^{(3)} + \frac{h^4}{4!}y_0^{(4)}$

$y_0' = x_0 - y_0 = -1$

$y_0'' = f'(x_0, y_0) = f_x + f_y y' = 1 - (x_0 - y_0) = 1 - x_0 + y_0 = 2$

$y_0''' = -1 + x_0 - y_0 = -2$

$y_0^{(4)} = 1 - x_0 + y_0 = 2$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0^{(3)} + \frac{h^4}{4!}y_0^{(4)}$$

$$= 1 + 0.1 * -1 + \frac{(0.1)^2}{2!} * 2 + \frac{(0.1)^3}{3!} * (-2) + \frac{(0.1)4}{4!} * 2 = 0.909675$$

i=1 $\longrightarrow$ $y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1^{(3)} + \frac{h^4}{4!}y_1^{(4)}$

$y_1' = 0.809675$

$y_1'' = 1.804675$

$y_1''' = -1.809675$

$y_1^{(4)} = 1.809675$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1^{(3)} + \frac{h^4}{4!}y_1^{(4)} = 0.837462$$

Also $y_3 = 0.781636$ $, and$ $y_4 = 0.740640$

Example (6): Use Taylor series method of order(3) to obtain the numerical solution y(2.1) of the initial value problem

$$y' = \frac{x-y}{x}, \quad y(2) = 2$$

Solution:

$f(x,y) = \frac{x-y}{x} = 1 - \frac{y}{x}$, $x_0=2$, $y_0=2$, $h=0.1$

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i^{(3)}$$

$$y(2.1) = y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0^{(3)}$$

$$y_0' = 1 - \frac{y_0}{x_0}$$

$$y_0'' = \frac{-1}{x_0} + \frac{2y_0}{x_0^2}$$

$$y_0''' = \frac{3}{x_0^2} - \frac{6y_0}{x_0^3}$$

y(2.1)=2.00238

Taylor Series Method Algorithm of order(k)

Input: x_0, y_0, n, f(x,y)

Step(1): compute $h = \frac{b-a}{n}$

Step(2): For i=1,2,…,n

Step(3): Set $x_{i+1} = x_i + h$

Step(4): Find all the derivatives at the point (x_i, y_i) i.e. : y' ,y'', y''', … $y^{(k)}$

Step(5): Evaluate $y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + ... + \frac{h^k}{k!} y_i^{(k)}$

Step(6): Print x_{i+1} and y_{i+1} for all i , then stop.