4- Runge-Kutta Methods

وضعت هذه الطريقة من قبل العالمين الالمانيين رانك وكوتا لتجنب حساب المشتقات العالية التي تتطلبها طريقة تيلر. وتعد طريقة رانك-كوتا من الرتبة الرابعة اكثرها شيوعا لسهولة استخدامها ورتبة الخطأ فيها عالية. تعرف صيغة رانك-كوتا من الرتبة الثانية Second Order Runge Kutta (RK-(2)) ذات خطأ البتر المحلي من الرتبة $O(h^3)$ بـ

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, y_i + h f(x_i, y_i))\right] \qquad (6)$$

$$= y_i + \frac{1}{2}[k_1 + k_2]$$

where $\quad k_1 = hf(x_i, y_i) \quad and \quad k_2 = hf(x_i + h, y_i + k_1)$

وتعرف صيغة رانك-كوتا من الرتبة الرابعة (Fourth Order Runge-Kutta (RK-4)) بـ

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (7)$$

where $\quad k_1 = hf(x_i, y_i)$

$$k_2 = hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

The local truncation error of RK-4 method is $O(h^5)$.

Example (7): Use RK-2 method to obtain the numerical solution y(1.01) and y(1.02) of the initial value problem

$$\frac{dy}{dx} = 1 + y^2 + x^3, \qquad y(1) = -4$$

Solution :

$$\text{h} = 0.01, x_0 = 1, \quad y_0 = -4, \; f(x,y) = 1 + y^2 + x^3$$

$$x_1 = 1.01, \quad y_1 = y_0 + \frac{h}{2}(k_1 + k_2) = -3.8254$$

$$k_1 = f(x_1, y_1) = (1 + {y_1}^2 + {x_1}^3) = 16.66$$

$$k_2 = f(x_1 + h, y_1 + k_1 h) = (1 + (y_1 + 0.1666)^2 + (x_1 + .01)^3) = 15.45$$

$$y_2 = y_1 + \frac{h}{2}(k_1 + k_2) = -3.8254 + \frac{0.01}{2}(16.66 + 15.45) = -3.6648$$

Example (8): Solve the I.V.P.

$$y' = x - y, \quad y(0) = 1$$

By using RK-4 method at the points x=0.1 , 0.2 , 0.3 , 0.4

Solution:

$f(x,y)=x-y$, $x_0=0$, $y_0=1$, $h=0.1$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_i, y_i)$

$$k_2 = hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

$$k_1 = hf(x_0, y_0) = 0.1(0-1) = -0.1$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1) = 0.1 f(0.05, 1 + \frac{1}{2}(-0.1)) = 0.1(0.05 - 0.95) = -0.09$$

$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2) = -0.0905$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = -0.08095$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.909675$$

Also

y_2=0.837461

y_3=0.781636

y_4=0.7406405

Runge-Kutta Method Algorithm of order four

Input: n , x_0 , y_0, f(x,y)

Step(1): compute $h = \frac{b-a}{n}$

Step(2): For i=1,2,…,n

Step(3): Find

$$k_1 = hf(x_{i-1}, y_{i-1})$$
$$k_2 = hf(x_{i-1} + \tfrac{1}{2}h, y_{i-1} + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_{i-1} + \tfrac{1}{2}h, y_{i-1} + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_{i-1} + h, y_{i-1} + k_3)$$

Step(4): Evaluate $y_i = y_{i-1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$x_{i+1} = x_i + h$

Step(5): Print x_i and y_i for all i , then stop.