2-Poisson’s equation

Consider Poisson’s equation

$$\nabla^2 u = g(x,\ y) \qquad (9)$$

let $g_{i,j} = g(x_i, y_j)$, for solving Poisson's equation (9) over the rectangular grid (square mesh) is

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} - h^2 g_{i,j} = 0 \qquad (10)$$

The local truncation error(L.T.E.) of formula (10) is

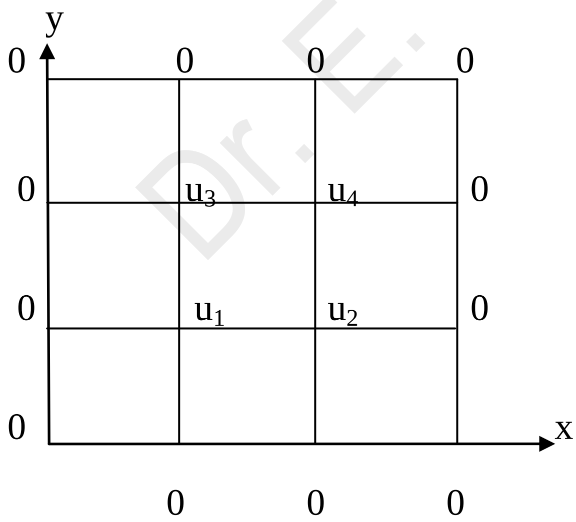
$$L.T.E. = \frac{h^2}{12}\left(u_{xxxx} + u_{yyyy}\right) + O(h^4)$$

Example: Solve the Poisson's equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides x=0=y, x=3=y with u=0 on the boundary and mesh length=1 .

Sol.:

The standard 5-point formula for the Poisson's equation is (from eq.(10))

$$u_{i-1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 g(ih,jh)$$
$$= -10(i^2+j^2+10) \qquad (11)$$



For (i=1,j=1) equation (11) gives $u_2+u_3-4u_1=-120$

For (i=2,j=1) equation (11) gives $u_1+u_4-4u_2=-150$

For (i=1,j=2) equation (11) gives $u_1+u_4-4u_3=-150$

For (i=2,j=2) equation (11) gives $u_2+u_3-4u_4=-180$

From these equations we have $u_2=u_3$ so these equations reduce to

$$-2u_1+u_2=-60$$
$$u_1-4u_2+u_4=-150$$
$$u_2-2u_4=-90$$

the solution of this system is $u_1=67.5$, $u_2=u_3=75$, $u_4=82.5$

Error Analysis (Poisson's equation)

Let $v_{i,j}$ be the true solution of eq.(9) at the point (x_i,y_j), let $e_{i,j}=v_{i,j}-u_{i,j}$, then

$$v_{i+1,j}+v_{i-1,j}+v_{i,j+1}+v_{i,j-1}-4v_{i,j}=h^2g_{i,j}+T_{i,j}$$
$$u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}=h^2g_{i,j}$$

the error satisfy
$$e_{i+1,j}+e_{i-1,j}+e_{i,j+1}+e_{i,j-1}-4e_{i,j}=T_{i,j}$$

where $T_{i,j}=\frac{h^4}{12}\left(u_{xxxx}+u_{yyyy}\right)+O(h^6)$ is the local truncation error.
Define a test function

$\Psi_{i,j}=e_{i,j}+ci(N+1-i)$,where c is constant

$$\Psi_{i+1,j}+\Psi_{i-1,j}+\Psi_{i,j+1}+\Psi_{i,j-1}-4\Psi_{,j}=T_{i,j}+c[(i+1)(N-i)+(i-1)(N-i+2)-2i(N+1-i)]$$
$$=T_{i,j}-2c$$

Now $T_{i,j}-2c\geq 0$ if we choose $c=-\frac{1}{2}\max_{i,j}\left|T_{i,j}\right|<0$

By the maximum value of $\Psi_{i,j}$ occurs on the boundary. Now $e_{i,j}=0$ on boundary

$$0\leq i(N+1-i)\leq 0\leq i(N+1-i)\leq\frac{(N+1)^2}{4}\quad, i=0,1,\ldots,N+1$$

So $\Psi_{i,j}\leq 0$ on boundary, $\Psi_{i,j}\leq 0$ inside by maximum principle

$$\Psi_{i,j}=e_{i,j}+ci(N+1-i)\leq 0$$

$$\longrightarrow\quad e_{i,j}\leq -ci(N+1-i)$$

$$\leq\frac{1}{2}T\frac{(N+1)^2}{4}$$, where $T=\max|T_{i,j}|$

Let $M_4 = \max_{over\ R\ over\ u_{4x}, u_{4y}} |fourth\ derivative\ of\ u|$

Then $T \leq \frac{h^4}{6} M_4$

$e_{i,j} \leq \frac{1}{2} * \frac{h^4}{6} M_4 * \frac{(N+1)^2}{4}$, where h(N+1)=1

$\leq \frac{h^2}{48} M_4$

$v_{i,j} - u_{i,j} \leq \frac{h^2}{48} M_4$

Similarly, choosing $\Psi_{i,j} = Di(N+1-i) - e_{i,j}$ *we can show that*

$-\frac{h^2}{48} M_4 \leq v_{i,j} - u_{i,j}$

Then $|e_{i,j}| = |v_{i,j} - u_{i,j}| \leq \frac{h^4}{48} M_4$

1- Helmholtz's Equation

Consider Helmholtz's equation

$$\nabla^2 u + f(x, y)u = g(x, y) \qquad (12)$$

Using the notation $f_{i,j} = f(x_i, y_j)$,for solving (12) over the rectangular grid is

$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - (4 - h^2 f_{i,j})\, u_{i,j} - h^2 g_{i,j} = 0$

Chapter Three

Parabolic Partial Differential Equations

As an example of parabolic differential equations, we consider the one-dimensional heat equation

$u_t(x, t) = c\, u_{xx}(x, t)$ (1)

where c is a positive constant and $0 \le x \le a$ and $t \ge 0$, with the initial condition

$u(x, 0) = f(x)$ for $t = 0$ and $0 \le x \le a$,

and the boundary conditions

$u(0, t) = g_1(t)$ for $x = 0$ and $t > 0$

$u(a, t) = g_2(t)$ for $x = a$ and $t > 0$

we cover the region with a rectangular grid with mesh length Δx or h in x-direction and Δt or k in the t-direction. Let $u_{i,j}$ is the approximation value of $u(x_i, t_j)$ where xi=ih , tj=jk.

The θ-method for eq.(1) is

$$\frac{u_{i,j} - u_{i,j-1}}{k} + Ac\left[\theta u_{i,j} + (1-\theta)u_{i,j-1}\right] = 0 \qquad , \quad \theta \epsilon [0,1]$$

Where A is the central second-order approximation of $-u_{xx}$,i.e.

$$Au_{i,j} = \frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h^2}$$

$$(I + \theta ckA)u_{i,j} = [I - (1-\theta)ckA]u_{i,j-1} \;, \quad \theta \epsilon [0,1] \qquad (2)$$

The following three choices of θ are popular

1- Forward Euler method ($\theta = 0$): eq.(2) is reduced to
   $u_{i,j} = [I - ckA]u_{i,j-1}$

   Which is explicit to compute the solution in each time level.

2- Backward Euler method ($\theta = 1$): this is implicit method written as
   $(I + ckA)u_{i,j} = u_{i,j-1}$

The method must invert a tridiagonal matrix to get the solution in each time level.

3- Crank-Nicholson method ($\theta = \frac{1}{2}$)

$$\left(I + \frac{1}{2}ckA\right)u_{i,j} = [I - \frac{1}{2}ckA]u_{i,j-1}$$

It requires to solve a tridiagonal system in each time level, this method is most popular because it is second-order in both space and time and unconditionally stable.

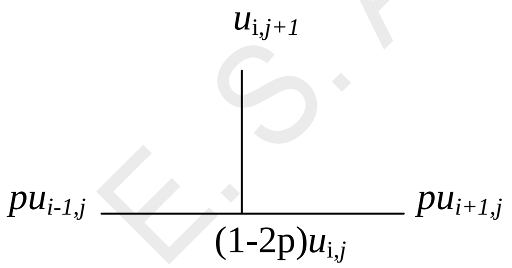
Classic Explicit Method

The difference formulas used for $u_t\,(x, t)$ and $u_{xx}\,(x, t)$ we have

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right)$$

$$u_{i,j+1} = pu_{i+1,j} + (1-2p)u_{i,j} + pu_{i-1,j} \qquad (3)$$

Where $p = \frac{ck}{h^2}$



If j=0 then R.H.S. of eq.(3) are known from the initial condition, thus we can calculate explicitly the values of the unknowns $u_{1,1}$, $u_{2,1}$,…, $u_{n,1}$. Then put j=1 in eq.(3) to calculate $u_{2,1}$, $u_{2,2}$,…, $u_{n,2}$, and so on .

Example: Use the Explicit method to solve the heat equation

$u_t\,(x, t) = u_{xx}\,(x, t)$ for $0 < x < 1$ and $0 < t < 0.2$,

with the initial condition

$u(x, 0) = f\,(x) = 4x - 4x^2$ for $t = 0$ and $0 \le x \le 1$,

and the boundary conditions

$u(0, t) = g_1(t) \equiv 0$ for $x = 0$ and $0 \le t \le 0.2$

$u(1, t) = g_2(t) \equiv 0$ for $x = 1$ and $0 \le t \le 0.2$

take h=0.2 and k=0.02.