Systems of Differential Equations

we consider the initial value problem

$$\frac{dx}{dt} = f(t, x, y)$$

(13)

$$\frac{dy}{dt} = g(t, x, y)$$

With $x(t_0) = x_0$, *and* $y(t_0) = y_0$.
A solution to (13) is a pair of differentiable functions $x(t)$ and $y(t)$ with the property that when t, $x(t)$, and $y(t)$ are substituted in $f(t, x, y)$ and $g(t, x, y)$, the result is equal to the derivative $x'(t)$ and $y'(t)$, respectively; that is,

$$x'(t) = f(t, x(t), y(t))$$
$$y'(t) = g(t, x(t), y(t))$$

with $x(t_0) = x_0$,*and* $y(t_0) = y_0$.

A numerical solution to (13) over the interval $a \le t \le b$ is found by using Euler's method

$$t_{i+1} = t_i + h,$$
$$x_{i+1} = x_i + h\,f(t_i, x_i, y_i) \qquad (14)$$
$$y_{i+1} = y_i + hg(t_i, x_i, y_i)$$

for $i = 0, 1, \ldots, M - 1$.The interval is divided into M subintervals of width $h = (b - a)/M$.

A higher-order method should be used to achieve a reasonable amount of accuracy. For example, the Runge-Kutta formulas of order 4 are

$$x_{i+1} = x_i + \frac{h}{6}\left(f_1 + 2f_2 + 2f_3 + f_4\right)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(g_1 + 2g_2 + 2g_3 + g_4\right) \qquad (15)$$

where $f_1 = f(t_i, x_i, y_i)$, $\quad g_1 = g(t_i, x_i, y_i)$

$f_2 = f(t_i + \tfrac{1}{2}h, x_i + \tfrac{1}{2}hf_1, y_i + \tfrac{1}{2}hg_1)$, $\quad g_2 = g(t_i + \tfrac{1}{2}h, x_i + \tfrac{1}{2}hf_1, y_i + \tfrac{1}{2}hg_1)$

$f_3 = (t_i + \tfrac{1}{2}h, x_i + \tfrac{1}{2}hf_2, y_i + \tfrac{1}{2}hg_2)$, $\quad g_3 = g(t_i + \tfrac{1}{2}h, x_i + \tfrac{1}{2}hf_2, y_i + \tfrac{1}{2}hg_2)$

$f_4 = f(t_i + h, x_i + hf_3, y_i + hg_3)$, $\quad g_4 = g(t_i + h, x_i + hf_3, y_i + hg_3)$

Example: Use the Runge-Kutta method of order 4 to compute the numerical solution of the system of differential equations

$$\frac{dx}{dt} = x + 2y \qquad\qquad x(0) = 6$$

with

$$\frac{dy}{dt} = 3x + 2y \qquad\qquad y(0) = 4$$

over the interval [0, 0.2] using 10 subintervals and the step size $h = 0.02$.
Where the solution is

$$x(t) = 4e^{4t} + 2e^{-t} ,$$
$$y(t) = 6e^{4t} - 2e^{-t} .$$

For the first point we have $t_1 = 0.02$, and the intermediate calculations required to compute x_1 and y_1 are

$f_1 = f(0.00, 6.0, 4.0) = 14$ $\qquad$ $g_1 = g(0.00, 6.0, 4.0) = 26$

$x_0 + (h/2)\, f_1 = 6.14$ $\qquad$ $y_0 + (h/2)\, g_1 = 4.26$

$f_2 = f(0.01, 6.14, 4.26) = 14.66$ $\qquad$ $g_2 = g(0.01, 6.14, 4.26) = 26.94$

$x_0 + (h/2)\, f_2 = 6.1466$ $\qquad$ $y_0 + (h/2) g_2 = 4.2694$

$f_3 = f(0.01, 6.1466, 4.2694) = 14.6854$

$g_3 = f(0.01, 6.1466, 4.2694) = 26.9786$

$x_0 + h\, f_3 = 6.293708$ $\qquad$ $y_0 + hg_3 = 4.539572$

$f_4 = f(0.02, 6.293708, 4.539572) = 15.372852$

$g_4 = f(0.02, 6.293708, 4.539572) = 27.960268.$

These values are used in the final computation:

$x_1 = 6 + (0.02/6)(14 + 2(14.66) + 2(14.6854) + 15.372852) = 6.29354551,$

$y_1 = 4 + (0.02/6)(26 + 2(26.94) + 2(26.9786) + 27.960268) = 4.53932490.$

similarly we have

$x_2 = 6.61562213$

$y_2 = 5.11948599$

and so on

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Higher-Order Differential Equations

Higher-order differential equations involve the higher derivatives $x''(t)$, $x'''(t)$, and so on. They arise in mathematical models for problems in physics and engineering. For example,

$$mx''(t) + cx'(t) + kx(t) = g(t)$$

represents a mechanical system in which a spring with spring constant k restores a displaced mass m. Damping is assumed to be proportional to the velocity, and the function $g(t)$ is an external force. It is often the case that the position $x(t_0)$ and velocity $x'(t_0)$ are known at a certain time t_0.
By solving for the second derivative, we can write a second-order initial value problem in the form

$$x''(t) = f(t, x(t), x'(t)) \quad \text{with} \quad x(t_0) = x_0 \ \text{ and } \ x'(t_0) = y_0 \qquad (16)$$

The second-order differential equation can be reformulated as a system of two first order equations if we use the substitution $x'(t) = y(t)$. Then $x''(t) = y'(t)$ and the differential equation in (16) becomes a system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = f(t, x, y)$$

with $x(t_0) = x_0$, *and* $y(t_0) = y_0$.

A numerical procedure such as the Runge-Kutta method can be used to solve the above system and will generate two sequences $\{x_i\}$ and $\{y_i\}$.

Example : Consider the second-order initial value problem

$$x''(t) + 4x'(t) + 5x(t) = 0 \quad \text{with} \quad x(0) = 3 \ \text{ and } \ x'(0) = -5.$$

(a) Write down the equivalent system of two first-order equations.
(b) Use the Runge-Kutta method to solve the reformulated problem over $[0, 5]$ using $M = 50$ subintervals of width $h = 0.1$.
(c) Compare the numerical solution with the true solution:

$$x(t) = 3e^{-2t}\cos(t) + e^{-2t}\sin(t).$$

Sol. (a) The differential equation has the form

$$x''(t) = f(t, x(t), x'(t)) = -4x'(t) - 5x(t).$$

(b) Using the substitution *x'(t) = y(t)*, we get the reformulated problem:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -5x - 4y$$

with *x(0) = 3, and y(0) = -5.*

(c) From the formulas in (15) we get the numerical solution then compare with the true solution

i	t_i	x_i	y_i
0	0.0	3.00000000	3.00000000
1	0.1	2.52564583	2.52565822
2	0.2	2.10402783	2.10404686
3	0.3	1.73506269	1.73508427
4	0.4	1.41653369	1.41655509
5	0.5	1.14488509	1.14490455
10	1.0	0.33324302	0.33324661
20	2.0	−0.00620684	−0.00621162
30	3.0	−0.00701079	−0.00701204
40	4.0	−0.00091163	−0.00091170
48	4.8	−0.00004972	−0.00004969
49	4.9	−0.00002348	−0.00002345
50	5.0	−0.00000493	−0.00000490

Finite-Difference Method

Methods involving difference quotient approximations for derivatives can be used for solving certain second-order boundary value problems. Consider the linear equation

$$x'' = p(t)x'(t) + q(t)x(t) + r(t) \tag{17}$$

over $[a, b]$ with $x(a) = \alpha$ and $x(b) = \beta$. Form a partition of $[a, b]$ using the points $a = t_0 < t_1 < \cdots < t_N = b$, where $h = (b - a)/N$ and $t_j = a + jh$ for $j = 0, 1, \ldots, N$. The central-difference formulas are used to approximate the derivatives

$$x'(t_j) = \frac{x(t_{j+1}) - x(t_{j-1})}{2h} + O(h^2) \tag{18}$$

$$x''(t_j) = \frac{x(t_{j+1}) - 2x(t_j) + x(t_{j-1})}{h^2} + O(h^2) \tag{19}$$

we replace each term $x(t_j)$ on the right side of (18) and (19) with x_j, and the resulting equations are substituted into (17) to obtain the relation

$$\frac{x(t_{j+1}) - 2x(t_j) + x(t_{j-1})}{h^2} + O(h^2) = p_j \frac{x(t_{j+1}) - x(t_{j-1})}{2h} + O(h^2) + q_j x_j + r_j$$

$$\frac{x(t_{j+1}) - 2x(t_j) + x(t_{j-1})}{h^2} = p_j \frac{x(t_{j+1}) - x(t_{j-1})}{2h} + q_j x_j + r_j \tag{20}$$

which is used to compute numerical approximations to the differential equation (17).
This is carried out by multiplying each side of (20) by h^2 and then collecting terms involving x_{j-1}, x_j, and x_{j+1} and arranging them in a system of linear equations:

$$\left(\frac{-h}{2} p_j - 1\right) x(t_{j-1}) + \left(2 + h^2 q_j\right) x_j + \left(\frac{h}{2} p_j - 1\right) x(t_{j+1}) = -h^2 r_j \tag{21}$$

for $j = 1, 2, \ldots, N-1$, where $x_0 = \alpha$ and $x_N = \beta$. The system in (21) has the familiar tridiagonal matrix :

$$\begin{bmatrix} 2+h^2q_1 & \frac{h}{2}p_1-1 & 0 & & & \\ \frac{-h}{2}p_2-1 & 2+h^2q_2 & \frac{h}{2}p_2-1 & & & \\ 0 & \frac{-h}{2}p_j-1 & 2+h^2q_j & \frac{h}{2}p_j-1 & & \\ & & \ddots & & & \\ & & \frac{-h}{2}p_{N-2}-1 & 2+h^2q_{N-2} & \frac{h}{2}p_{N-2}-1 & \\ & & & \frac{-h}{2}p_{N-1}-1 & 2+h^2q_{N-1} & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} -h^2r_1+e_0 \\ -h^2r_2 \\ \vdots \\ -h^2r_j \\ \vdots \\ -h^2r_{N-2} \\ -h^2r_{N-1}+e_N \end{bmatrix}$$

Where $e_0 = (\frac{h}{2}p_1+1)\alpha$ and $e_N = (\frac{-h}{2}p_{N-1}+1)\beta$

When computations with step size h are used, the numerical approximation to the solution is a set of discrete points $\{(t_j, x_j)\}$; if the analytic solution $x(t_j)$ is known, we can compare x_j and $x(t_j)$.

Example : Solve the boundary value problem

$$x'' = \frac{2t}{1+t^2}\,x'(t) - \frac{2}{1+t^2}x(t) + 1$$

with $x(0) = 1.25$ and $x(4) = -0.95$ over the interval [0, 4].

The functions p, q, and r are $p(t) = 2t/(1+t^2)$, $q(t) = -2/(1+t^2)$, and $r(t) = 1$, respectively. The finite-difference method is used to construct numerical solutions $\{x_j\}$ using the above system of equations
the analytic solution:

$x(t) = 1.25 + 0.486089652t - 2.25t^2 + 2t\arctan(t) - 1/2\ln(1+t^2) + 1/2t^2\ln(1+t^2)$.

t_j	$x_{j,1}$ $h = 0.2$	$x_{j,2}$ $h = 0.1$	$x_{j,3}$ $h = 0.05$	$x_{j,4}$ $h = 0.025$	$x(t_j)$ exact
0.0	1.250000	1.250000	1.250000	1.250000	1.250000
0.2	1.314503	1.316646	1.317174	1.317306	1.317350
0.4	1.320607	1.325045	1.326141	1.326414	1.326505
0.6	1.272755	1.279533	1.281206	1.281623	1.281762
0.8	1.177399	1.186438	1.188670	1.189227	1.189412
1.0	1.042106	1.053226	1.055973	1.056658	1.056886
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.					
4.0	−0.950000	−0.950000	−0.950000	−0.950000	−0.950000

H.W.: solve for h=1?

Chapter Two
Numerical Solution of Partial-Differential Equations and Elliptic equation

Many problems in applied science, physics, and engineering are modeled mathematically with partial differential equations. A differential equation involving more than one independent variable is called a ***partial differential equation*** (PDE).In this chapter we will study finite differences methods which are based on formulas for approximating the first and second derivatives of a function. We start by classifying the three types of equations under investigation ,a partial differential equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = f\,(x, y, u, u_x\,, u_y) \qquad (1)$$

where A, B, and C are constants, u is called ***quasilinear***. There are three types of quasilinear equations:

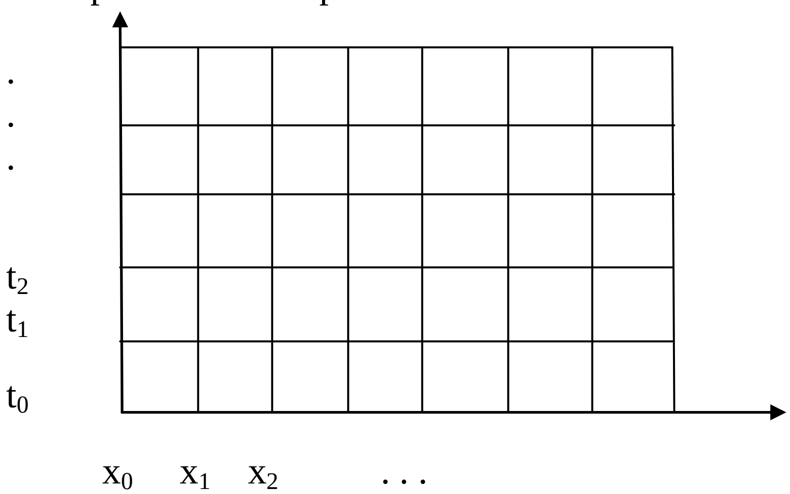
(1) If $B^2 - 4AC < 0$, the equation is called ***elliptic***.
(2) If $B^2 - 4AC = 0$, the equation is called ***parabolic***.
(3) If $B^2 - 4AC > 0$, the equation is called ***hyperbolic***.

Finite differences approximation to partial derivative

To find a numerical solution u to equation (1) with finite difference methods, we first need to divide the (x,y)-plane into a network of rectangles of sides $\Delta x = h$ and $\Delta t = k$ by drawing the set of lines

$x = i\,h$, $i = 0,1,2, \ldots$
$y = jk$, $j = 0,1,2, \ldots$

the points of intersection of these families of lines are called grid points or lattic points or mesh points.



The finite differences formulas for approximating $u_x(x,y)$, $u_y(x,y)$, $u_{xx}(x,y)$ and $u_{yy}(x,y)$ are

$$u_x\,(x,y) = \frac{u(x+h,y) - u(x,y)}{h} + O(h)$$

$$u_x\,(x,y) = \frac{u(x,y) - u(x-h,y)}{h} + O(h)$$

$$u_x\,(x,y) = \frac{u(x+h,y) - u(x-h,y)}{2h} + O(h^2)$$

$$u_y\,(x,y) = \frac{u(x,y+k) - u(x,y)}{k} + O(k)$$

$$u_y\,(x,y) = \frac{u(x,y) - u(x,y-k)}{k} + O(k)$$

$$u_x\,(x,y) = \frac{u(x,y+k) - u(x,y-k)}{2k} + O(k^2)$$

$$u_{xx}\,(x,y) = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + O(h^2)$$

$$u_{yy}\,(x,y) = \frac{u(x,y+k) - 2u(x,y) + u(x,y-k)}{k^2} + O(k^2)$$