Numerical Solution of and Elliptic Equations

We consider the Laplace, Poisson, and Helmholtz equations as examples of elliptic partial differential equations, we can write the Laplace, Poisson, and Helmholtz equations in the following forms:

(1) $\nabla^2 u = u_{xx} + y_{yy} = 0$ Laplace's equation
(2) $\nabla^2 u = g(x, y)$ Poisson's equation
(3) $\nabla^2 u + f(x, y)u = g(x, y)$ Helmholtz's equation

the boundary values for the function u are known at all points on the sides of a rectangular region R in the plane. In this case, each of these equations can be solved by the numerical technique known as the finite-difference method.

1- Laplacian Difference Equation

$$\nabla^2 u = \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2} + O(h^2) + \frac{u(x,y+k)-2u(x,y)+u(x,y-k)}{k^2} + O(k^2) \qquad (2)$$

Assume that the rectangle $R = \{(x, y) : 0 \le x \le a, 0 \le y \le b$, where $b/a = m/n\}$ is subdivided into $n - 1 \times m - 1$ squares with side h (i.e., $a = nh$ and $b = mh$), and all interior grid points $(x, y) = (x_i, y_j)$ for $i = 2, \ldots, n - 1$ and $j = 2, \ldots, m - 1$. The grid points are uniformly spaced:

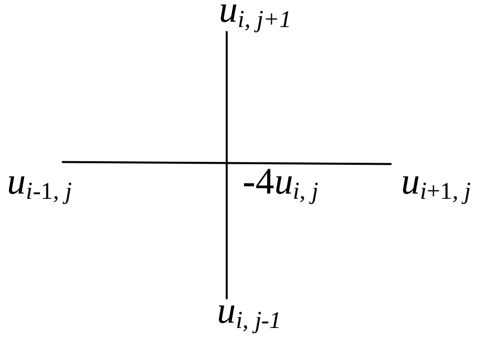
$x_{i+1} = x_i + h$, $x_{i-1} = x_i - h$, $y_{i+1} = y_i + h$, and $y_{i-1} = y_i - h$

Using the approximation $u_{i,j}$ for $u(x_i, y_j)$, equation (2) can be written in the form

$$\nabla^2 u = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) \qquad (3)$$

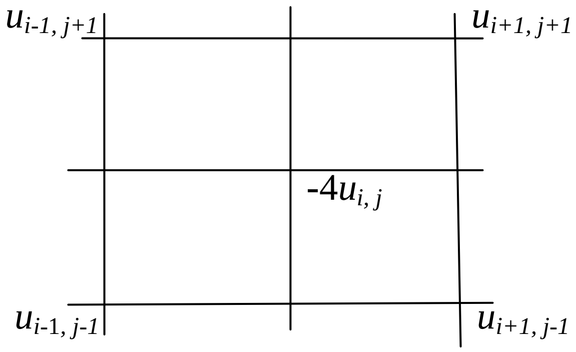
Eq.(3) is known as the ***standard five-point difference formula*** for Laplace's equation. This formula relates the function value $u_{i,j}$ to its four neighboring values $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$, and $u_{i,j-1}$



sometimes we use a formula similar to eq.(3) which given by

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \qquad (4)$$

Eq.(4) is known as the ***diagonal five-point difference formula*** for Laplace's equation.



The values $u(x, y)$ are known at the following boundary grid points:

$u(x_1, y_j) = u_{1,j}$ for $2 \le j \le m-1$ (on the left)
$u(x_i, y_1) = u_{i,1}$ for $2 \le i \le n-1$ (on the bottom)
$u(x_n, y_j) = u_{n,j}$ for $2 \le j \le m-1$ (on the right)
$u(x_i, y_m) = u_{i,m}$ for $2 \le i \le n-1$ (on the top)

Then applying the Laplacian computational formula (3) at each of the interior points of R will create a linear system of $(n-2)$ equations in $(n-2)$ unknowns, which is solved to obtain approximations to $u(x, y)$ at the interior points of R.

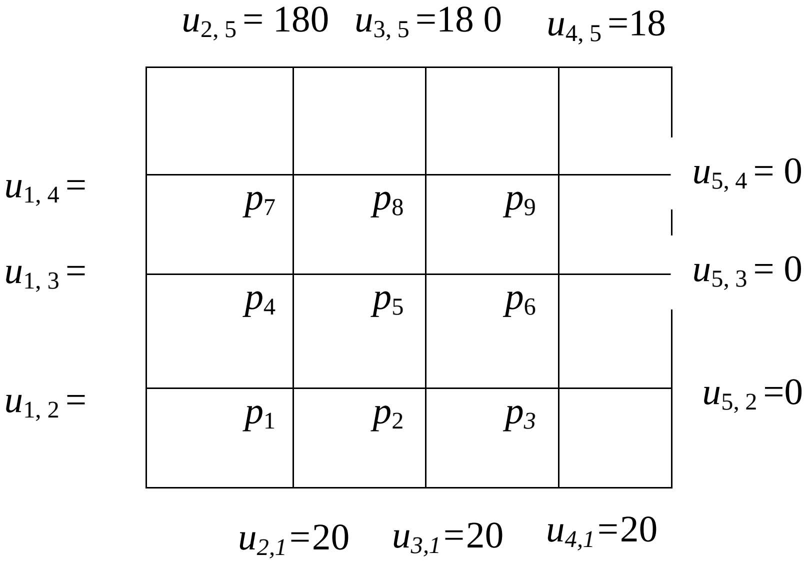
The local truncation error(L.T.E.) of formula (3) is

$$L.T.E. = \frac{h^2}{12}\left(u_{xxxx} + u_{yyyy}\right) + O(h^4)$$

Example : Find an approximate solution to Laplace's equation $\nabla^2 u = 0$
in the rectangle $R = \{(x, y) : 0 \le x \le 4,\ 0 \le y \le 4\}$, where $u(x, y)$ denotes the temperature at the point (x, y) and the boundary values are
$u(x, 0) = 20$ and $u(x, 4) = 180$ for $0 < x < 4$
$u(0, y) = 80$ and $u(4, y) = 0$ for $0 < y < 4$

Sol.

Suppose that the region is a square that n=m=5 and that the unknown values of $u(x_i, y_j)$ at the nine interior grid points are labeled $p_1, p_2, \ldots, p_9$ and positioned in the grid



Applying formula (3) in this case, the linear system $AP = B$ is

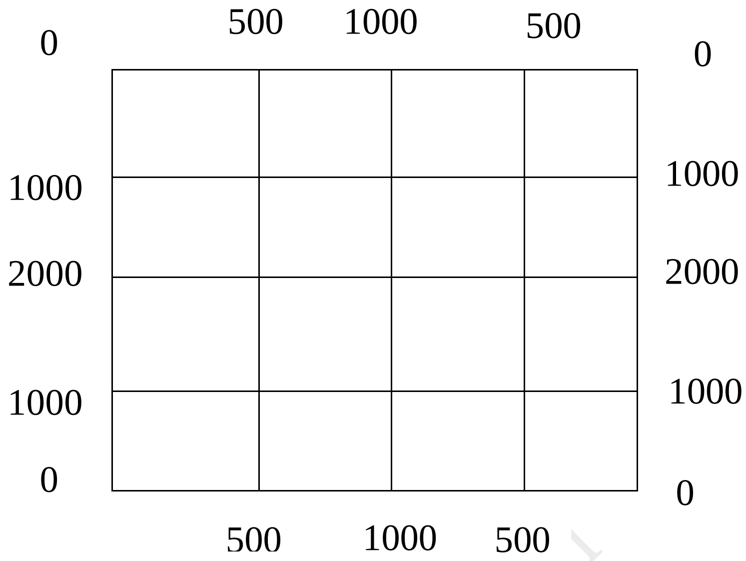
$$
\begin{aligned}
-4p_1 + p_2 + p_4 &= -100 \\
p_1 - 4p_2 + p_3 + p_5 &= -20 \\
p_2 - 4p_3 + p_6 &= -20 \\
p_1 - 4p_4 + p_5 + p_7 &= -80 \\
p_2 + p_4 - 4p_5 + p_6 + p_8 &= 0 \\
p_3 + p_5 - 4p_6 + p_9 &= 0 \\
p_4 - 4p_7 + p_8 &= -260 \\
p_5 + p_7 - 4p_8 + p_9 &= -180 \\
p_6 + p_8 - 4p_9 &= -180
\end{aligned}
$$

The solution vector P can be obtained by Gaussian elimination (or more efficient schemes can be devised. The temperatures at the interior grid points are expressed in vector form

$P = [p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \quad p_8 \quad p_9]^t$

$= [55.7143 \quad 43.2143 \quad 27.1429 \quad 79.6429 \quad 70 \quad 45.3571 \quad 112.857 \quad 111.786 \quad 84.2857]^t$

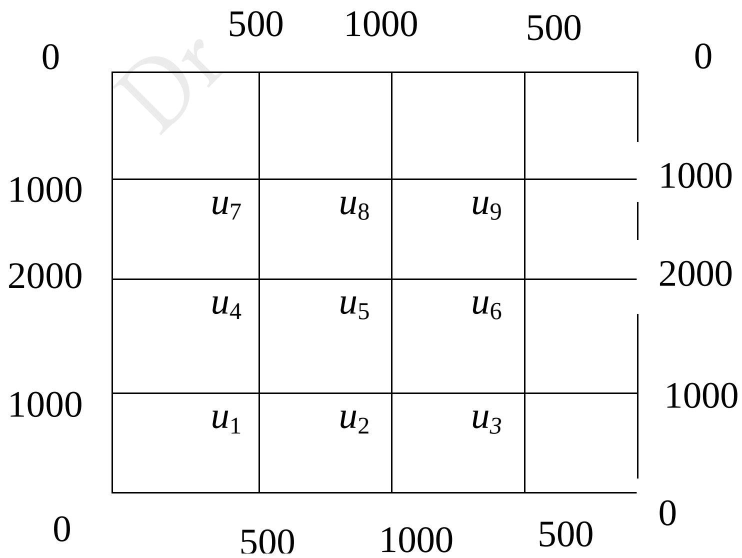
Example : Solve the Laplace equation $\nabla^2 u = 0$ for the following square mesh with boundary values given in figure



Sol.: The Laplace equation $\nabla^2 u = 0$ is an elliptic equation

$$\nabla^2 u = 0 \quad \rightarrow \quad u_{xx}+u_{yy}=0$$

Let $u_1, u_2, \ldots , u_9$ be the values of u at the interior mesh points as shown in the following figure:



from mesh the boundary values of u are symmetrically

$u_1 = u_7$, $u_2 = u_8$, $u_3 = u_9$, and $u_3 = u_1$, $u_6 = u_4$, $u_9 = u_7$

so it is sufficient to find the values of u_1 , u_2 , u_4 *and* u_5

Now, we find their initial values

$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500$ (standard formula)

$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125$ (diagonal formula)

$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) = 1188$ (standard formula)

$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) = 1438$ (standard formula)

We use the Gauss-seidel method

$$u_1^{(n+1)} = \frac{1}{4}\left(1000 + u_2^{(n)} + 500 + u_4^{(n)}\right)$$
$$u_2^{(n+1)} = \frac{1}{4}\left(u_1^{(n+1)} + u_1^{(n)} + 1000 + u_5^{(n)}\right)$$
$$u_4^{(n+1)} = \frac{1}{4}\left(2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n)}\right)$$
$$u_5^{(n+1)} = \frac{1}{4}\left(u_4^{(n+1)} + u_4^{(n)} + u_2^{(n+1)} + u_2^{(n)}\right)$$

First iteration ,put n=0 we have

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) = 1032$$
$$u_2^{(1)} = \frac{1}{4}(1032 + 1125 + 1000 + 1500) = 1164$$
$$u_4^{(1)} = \frac{1}{4}(2000 + 1500 + 1032 + 1125) = 1414$$
$$u_5^{(1)} = \frac{1}{4}(1414 + 1438 + 1164 + 1188) = 1301$$

second iteration ,put n=1 we have

$u_1^{(2)}$ =1020 , $u_2^{(2)}$ =1088 , $u_4^{(2)}$ =1338 , $u_5^{(2)} = 1251$

And so on …, we see that there is negligible difference between the values obtained in 11th and 12th iterations

$u_1 = 939$, $u_2 = 1001$, u_4 =1251 , and u_5 =1126

Derivative Boundary Conditions

There are three types of the boundary conditions with Laplace's equation

(i) Dirichlet → u=f(x,y) on ∂R

(ii) Nuemann → $\frac{\partial u}{\partial n} = g(x, y)$ on ∂R

(iii) Mixed → $\alpha(x, y) + \beta(x, y)\frac{\partial u}{\partial n} = \gamma(x, y)$ on ∂R

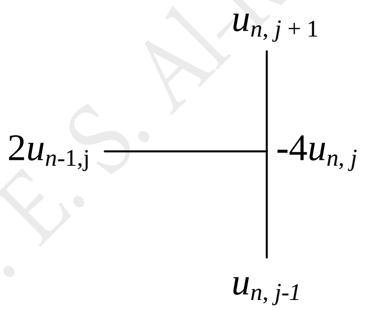
where $\alpha(x, y)$ & $\beta(x, y) \geq 0$ on ∂R

The Neumann boundary conditions specify the directional derivative of $u(x,y)$ normal to an edge.

$$\frac{\partial u(x, y)}{\partial n} = 0$$

Suppose that $x = x_n$ is held fixed and that we are considering the right edge $x = a$ of the rectangle $R = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$. The normal boundary condition to be used along this edge is

$$\frac{\partial u(x_n, y_j)}{\partial x} = u_x(x_n, y_j) = 0$$



(right edge)

Then the Laplace difference equation for the point (x_n, y_j) is

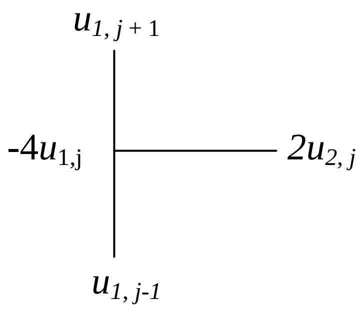
$$u_{n+1, j} + u_{n-1, j} + u_{n, j+1} + u_{n, j-1} - 4u_{n, j} = 0 \quad (5)$$

The value $u_{n+1, j}$ is unknown, because it lies outside the region R. However, we can use the numerical differentiation formula

$$\frac{u_{n+1,j} - u_{n-1,j}}{2h} \approx u_x(x_n, y_j) = 0$$

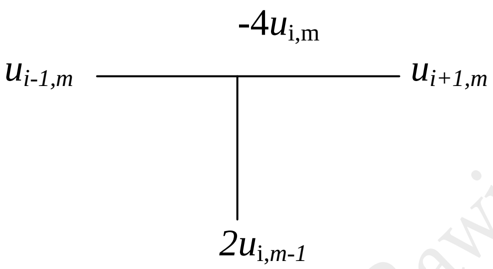
and obtain the approximation $u_{n+1, j} \approx u_{n-1, j}$, which has order of accuracy $O(h^2)$. When this approximation is used in (5), the result is

$$2u_{n-1, j} + u_{n, j+1} + u_{n, j-1} - 4u_{n, j} = 0$$



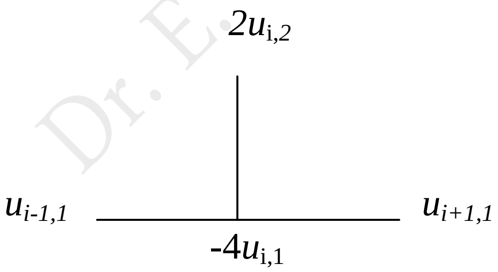
(left edge)

$$2u_{2,j} + u_{1,j-1} + u_{1,j+1} - 4u_{1,j} = 0 \qquad (6)$$



(top edge)

$$2u_{i,m-1} + u_{i-1,m} + u_{i+1,m} - 4u_{i,m} = 0 \qquad (7)$$



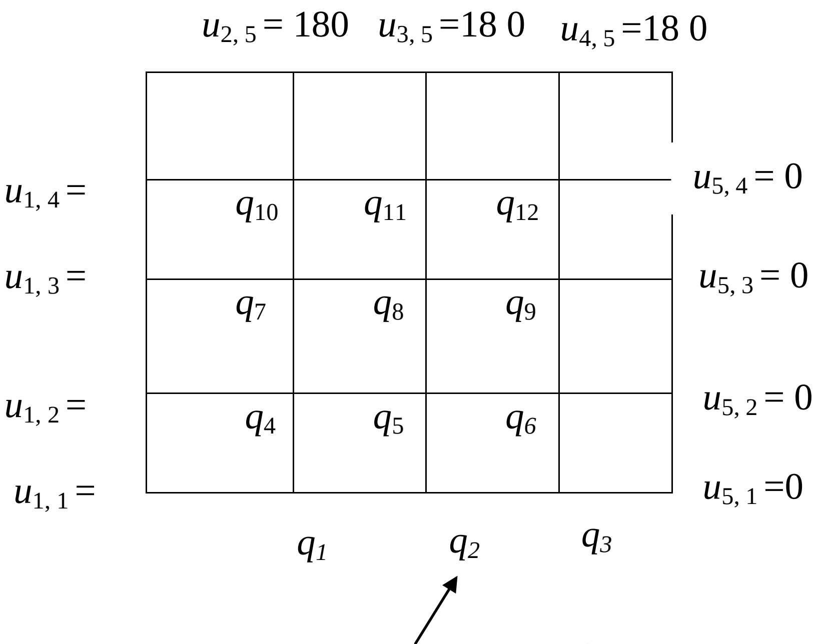
(bottom edge)

$$2u_{i,2} + u_{i-1,1} + u_{i+1,1} - 4u_{i,1} = 0 \qquad (8)$$

Example : Find an approximate solution to Laplace's equation $\nabla^2 u = 0$ in the rectangle $R = \{(x, y) : 0 \le x \le 4, 0 \le y \le 4\}$, where $u(x, y)$ denotes the temperature at the point (x, y) and the boundary values are

$u(x, 4) = 180$ for $0 < x < 4$
$u_y(x, 0) = 0$ for $0 < x < 4$
$u(0, y) = 80$ for $0 \le y < 4$
$u(4, y) = 0$ for $0 \le y < 4$

Sol.:



$$\left.\frac{\partial u(x,y)}{\partial y}\right|_{y=0} = 0$$

The Neumann computational formula (8) is applied at the boundary points q_1, q_2, and q_3, and the Laplace computational stencil (3) is applied at the other points q_4, q_5, . . ., q_{12}. The result is a linear system $AQ = B$ involving 12 equations in 12 unknowns:

$$-4q_1 + q_2 + 2q_4 = -80$$
$$q_1 - 4q_2 + q_3 + 2q_5 = 0$$
$$q_2 - 4q_3 + 2q_6 = 0$$
$$q_1 - 4q_4 + q_5 + q_7 = -80$$
$$q_2 + q_4 - 4q_5 + q_6 + q_8 = 0$$
$$q_3 + q_5 - 4q_6 + q_9 = 0$$
$$q_4 - 4q_7 + q_8 + q_{10} = -80$$
$$q_5 + q_7 - 4q_8 + q_9 + q_{11} = 0$$
$$q_6 + q_8 - 4q_9 + q_{12} = 0$$
$$q_7 - 4q_{10} + q_{11} = -260$$
$$q_8 + q_{10} - 4q_{11} + q_{12} = -180$$
$$q_9 + q_{11} - 4q_{12} = -180$$

The solution vector Q can be obtained by Gaussian elimination. The temperatures at the interior grid points and along the lower edge are expressed in vector form as

$Q = [q_1\ \ q_2\ \ q_3\ \ q_4\ \ q_5\ \ q_6\ \ q_7\ \ q_8\ \ q_9\ \ q_{10}\ \ q_{11}\ \ q_{12}]^t$

$= [71.8218\ \ 56.8543\ \ 32.2342\ \ 75.2165\ \ 61.6806\ \ 36.0412\ \ 87.3636\ \ 78.6103\ \ 50.2502\ \ 115.628\ \ 115.147\ \ 86.3492]^t$