Exponential Finite Difference Method

This method first introduced by Bathacharya (1985) to the unstable state of one-dimensional thermal conductivity of the Cartesian coordinates, and then developed the method algorithm by Robert to solve the one-dimensional propagation equation in cylindrical coordinates . It was applied to two- and three-dimensional .

To derive this method we use eq.(1) and let F(u) be a continuous function has derivative then multiply eq.(1) by $F'(u)$ we have

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = cF'(u)(\frac{\partial^2 u}{\partial x^2})$$

$$\frac{\partial F}{\partial t} = cF'(u)(\frac{\partial^2 u}{\partial x^2})$$

$$\frac{F(u_{i,j+1}) - F(u_{i,j})}{k} = cF'(u_{i,j})(\frac{\partial^2 u}{\partial x^2})_{i,j}$$

$$F(u_{i,j+1}) = F(u_{i,j}) + ckF'(u_{i,j})(\frac{\partial^2 u}{\partial x^2})_{i,j}$$

Now let F(u)=ln(u)

$$u_{i,j+1} = u_{i,j} * e^{ck\frac{1}{u_{i,j}}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}}$$

$$= u_{i,j} * e^{\frac{ck}{h^2}(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{u_{i,j}})} \qquad (8)$$

Notes:

1- The L.T.E. of this method is O(k)+O(h^2).

2- The Exponential F.D.M. is unconditionally stable.

Chapter Four

Hyperbolic Partial Differential Equations of first order and second order

First order equation:

Consider the equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad , \qquad x \geq 0 \ , \quad t \geq 0 \qquad (1)$$

Where a is a constant. The general solution of (1) is u(x,t)=Φ(x-at), where Φ is an arbitrary functions. The initial and boundary conditions are

u(x,0)=f(x)

u(0,t)=g(x)

Simple Explicit Method

The difference formulas used for $u_t\ (x, t)$ and $u_x\ (x, t)$ we have

$$\frac{u_{i,j+1} - u_{i,j}}{k} + a\left(\frac{u_{i,j} - u_{i-1,j}}{h}\right) = 0$$

$$u_{i,j+1} = (1-p)u_{i,j} + pu_{i-1,j} \qquad (2)$$

Where $p = \frac{ak}{h}$

Local Truncation Error of Simple Explicit Method:

$$\frac{v_{i,j+1} - v_{i,j}}{k} + a\left(\frac{v_{i,j} - v_{i-1,j}}{h}\right) = T_{i,j}$$

Where $v_{i,j}$ is the true solution of P.D.E.at the point (x_i,t_j)=(ih,jk) and $T_{i,j}$ is the L.T.E. expand in a Taylor's series about the point (x_i,t_j). we assume that sufficiently many derivatives of v exist.

$$v_{i,j+1} = v + kv_t + \frac{1}{2}k^2v_{tt} + \frac{1}{6}k^3v_{ttt} + \cdots$$

$$v_{i-1,j} = v - hv_x + \frac{1}{2}h^2v_{xx} - \frac{1}{6}h^3v_{xxx} + \cdots$$

Where everything on the right hand side is evaluation at (x_i,t_j)

$$T_{i,j} = v_t + \frac{1}{2}kv_{tt} + \frac{1}{6}k^2v_{ttt} + \cdots + a\{v_x - \frac{1}{2}hv_{xx} + \frac{1}{6}h^2v_{xxx} - \cdots\}$$

$$= \frac{1}{2}kv_{tt} - \frac{1}{2}ahv_{xx} + \frac{1}{6}k^2v_{ttt} + \frac{1}{6}ah^2v_{xxx} + \cdots$$

$$= \frac{1}{2} a(ak - h)v_{xx} + \frac{1}{6} a(h^2 - a^2k^2)v_{xxx} + \cdots$$

If p=1 then ak=h and the numerical solution is identical to the analytic solution. All coefficients in L.T.E. vanish in this case.

Numerical Stability:

We investigate the stability of this scheme using the Von Neumann criterion. Let $u_{i,j} = \vartheta(t)e^{i\alpha nh}$, from the explicit scheme we get

$$\frac{\vartheta(t+k)e^{i\alpha nh} - \vartheta(t)e^{i\alpha nh}}{k} + a\frac{\vartheta(t)e^{i\alpha nh} - \vartheta(t)e^{i\alpha(n-1)h})}{h^2} = 0$$

$$\vartheta(t+k) - \vartheta(t) + \frac{ak}{h}\vartheta(t) - \frac{ak}{h}\vartheta(t)e^{-i\alpha h} = 0$$

$$\frac{\vartheta(t+k)}{\vartheta(t)} = 1 - \frac{ak}{h}\left(1 - e^{-i\alpha h}\right) = \xi$$

The von Neumann stability condition is

$$\left|\frac{\vartheta(t+k)}{\vartheta(t)}\right| = |\xi| \le 1$$

$$\Longleftrightarrow \quad |\xi| = \left|1 - p\left(1 - e^{-i\alpha h}\right)\right| \quad , \quad p = \frac{ak}{h}$$

$$\Longleftrightarrow \quad = |1 - p(1 - cos\alpha h) - ipsin\alpha h|$$

$$\Longleftrightarrow \quad |\xi|^2 = 2p^2 + 1 - 2p^2cos\alpha h - 2p + 2pcos\alpha h$$

$$= 2p^2(1 - cos\alpha h) - 2p(1 - cos\alpha h) + 1$$

$$= 2p(1 - p)(1 - cos\alpha h) + 1$$

Since $1 - cos\alpha h \ge 0$, $\forall \alpha, h$. For $|\xi|^2 \le 1$, we must have p(p-1)<0

then $0 < p \le 1$, therefore this method is stable provided $\frac{ak}{h} \le 1 \rightarrow k \le \frac{h}{a}$.

Crank-Nicholson method

In this method we use the following approximation to u_x at the point (x_i,t_j)

$$u_x = \frac{1}{2}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h}\right)$$

therefore the Crank-Nicolson method for solving eq.(1) is

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{a}{4h}\left(u_{i+1,j} - u_{i-1,j} + u_{i+1,j+1} - u_{i-1,j+1}\right) \qquad (3)$$

If we solve the problem in the region $\{(x, y) : 0 \le x \le 1, t \ge 0\}$ then we have N linear equations for N+1 unknowns at the (j+1) time step.

Note: To find L.T.E. we expand in a Taylor's series about the mid-point of the point $(x_i, t_j + \frac{1}{2}k)$. The Von Neumann shows that there is unconditionally stable.

Second order equation:

Consider the wave equation

$$u_{tt}(x, t) = c^2 u_{xx}(x, t) \qquad (4)$$

for $0 < x < a$ and $t > 0$, with the initial condition
$u(x, 0) = f(x)$ for $t = 0$ and $u_t|_{(x,0)} = g(x)$

and the boundary conditions
$u(0, t) = g_1(t)$ for $x = 0$ and $t > 0$
$u(a, t) = g_2(t)$ for $x = a$ and $t > 0$

Explicit method:

Let us consider a simple hyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad t > 0 \quad , \quad 0 \le x \le 1 \quad , \quad c=1 \qquad (5)$$

with the initial condition
$u(x, 0) = f(x)$ and $u_t|_{(x,0)} = g(x)$ for $t = 0$, $0 \le x \le 1$

and the boundary conditions
$u(0, t) = g_1(t)$, $u(1, t) = g_2(t)$, $t > 0$

the central difference approximation for eq.(5) is

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$u_{i,j+1} = -u_{i,j-1} + 2u_{i,j} + p(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$ where $p=\frac{k^2}{h^2}$

$= -u_{i,j-1} + 2(1-p)u_{i,j} + p(u_{i+1,j} + u_{i-1,j})$ (6)

Now u_t=g(x) then

$$\frac{u_{i,j+1} - u_{i,j}}{k} = g(x)$$

$$u_{i,j+1} = u_{i,j} + kg(x) \quad at \quad t = 0$$

For t=0 i.e. j=1 ⟶ $u_{i,2} = u_{i,1} + kg(x)$ (7)

and we have from the I.C.

u=f(x) at t=o

$u_{i,j} = f(x) \quad at \quad t = 0$

$u_{i,1} = f(x) \quad at \quad t = 0$ (8)

Combining eq. (7) and eq.(8) we have

$u_{i,2} = f(x) + kg(x)$ (9)

Also $u_{1,j} = g_1(t)$, $u_{n,j} = g_2(t)$

Local Truncation Error

Let $v_{i,j}$ is the true solution of the wave equation at the point (x_i,t_j)=(ih,jk) and from eq.(5)we have

$$\frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{k^2} = v_{tt} + \frac{k^2}{12} v_{4t} + \cdots$$

$$\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} = v_{xx} + \frac{h^2}{12} v_{4x} + \cdots$$

Now

$$T_{i,j} = v_{tt} + \frac{k^2}{12} v_{4t} + \cdots - \left(v_{xx} + \frac{h^2}{12} v_{4x} + \cdots\right)$$

$$= \frac{k^2}{12} v_{4t} + \cdots - \frac{h^2}{12} v_{4x} - \cdots = O(k^2) + O(h^2)$$