Numerical Stability:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0$$

Let $u_{i,j} = \vartheta(t)e^{i\alpha nh}$, from the explicit scheme i.e.: eq.(6)

$u_{i,j+1} + u_{i,j-1} + 2(p-1)u_{i,j} - p\left(u_{i+1,j} + u_{i-1,j}\right) = 0$ where p= $\frac{k^2}{h^2}$

we have the explicit scheme is stable if $p \leq 1 \rightarrow k \leq h$.

Example: Use the finite differences method to solve the wave equation for a vibrating string:

$$u_{tt}(x, t) = 4u_{xx}(x, t) \quad \text{for} \quad 0 < x < 1 \text{ and } 0 < t < 0.5$$

with the initial and boundary conditions

$$u(x,0) = f(x) = \begin{Bmatrix} x & for & 0 \leq x \leq \frac{3}{5} \\ 1.5 - 1.5x & for & \frac{3}{5} < x \leq 1 \end{Bmatrix}$$

$u_t(x,0) = g(x) = 0 \qquad for \qquad 0 < x < 1$

$u(0,t)=0$ and $u(1,t)=0$ for $0 \leq t \leq 0.5$

choose h=0.1 and k=0.05

Sol.: Since c=2 and p=1 then fom eq.(6) we have

$$u_{i,j+1} = -u_{i,j-1} + u_{i+1,j} + u_{i-1,j}$$

First level:

$u_{i,1}$: 0 0.1 0.2 0.3 0.4 0.5 0.6 0.45 0.3 0.15 0

Second level:

$u_{i,2}$: 0 0.1 0.2 0.3 0.4 0.5 0.6 0.45 0.3 0.15 0

Third level:

$u_{i,3}$: 0 0.1 0.2 0.3 0.4 . . .

and so on.

Crank-Nicholson method

To solve eq.(5) by using this method we have

$$u_{tt} = c^2 u_{xx}$$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{c^2}{2}\left(\frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right)$$

Let $p = \frac{c^2k^2}{h^2}$ then we have

$$pu_{i+1,j+1} - 2(1+p)u_{i,j+1} + pu_{i-1,j+1} = -4u_{i,j} + p\left(u_{i-1,j-1} + u_{i+1,j-1}\right) + 2(1+p)u_{i,j-1} \qquad (10)$$

Example: Solve the problem of $u_{tt}(x, t) = u_{xx}(x, t)$ for $0 \le x \le 1$ with the initial and boundary conditions

$$u(x,0) = \frac{1}{2}\mathrm{x}(1-\mathrm{x}) \text{ and } u_t(x,0) = 0 \quad for \quad 0 \le x \le 1$$

$u(0,t)=0$ and $u(1,t)=0$ for $t \ge 0$

using Crank-Nicholson scheme up to third level taking h=k=0.1

Sol.:

The grid values at the first level are:

0 0.045 0.08 0.105 0.012 0.125 0.12 0.105 0.08 0.045 0

From $u_t(x,0) = 0$ we have $u_{i,j+1} = u_{i,j}$, the grid values at the second level are:

0 0.045 0.08 0.105 0.012 0.125 0.12 0.105 0.08 0.045 0

For the next level grid values, setting j=2 in eq.(10) we have (p=1)

$$u_{i-1,3} - 4u_{i,3} + u_{i+1,3} = -4u_{i,2} - \left(u_{i-1,1} - 4u_{i,1} + u_{i+1,1}\right), i = 2,3, \ldots ,10$$

We have a system of equations and we use the Gauss-Seidel process to find the solution.

Chapter Five

Numerical Analysis of Delay Differential Equations

Introduction to Delay Differential Equations

Delay Differential Equations (DDEs) are a special class of more general functional equations, such as integro-differential equations. A more general type of differential equation, often called a functional differential equation, is one in which the unknown function occurs with various different argument.

The differential equation :

$$y'(t) = f(t, y(t)) \qquad (1)$$

The functional differential equation :

$$y'(t) = f(t, y(t), y(\zeta_j(t))), \quad 0 \le t < \infty \ , \ j = 1,2,\ldots,m \qquad (2)$$

With initial function y(t)=φ(t) for $t \le 0$. In delay differential equation, y'(t) does not depend on the y(t) at very instant, but really at some other timey(t-τ) or later time y(t+τ) where τ constant. Time delays are natural component of the dynamic processes of biology, ecology, physiology, economics, and mechanics, other literatures write equation (2) in the following form

$$y'(t) = f(t, y(t), y(\lambda_j t)), \quad 0 \le t < \infty \ , \ j = 1,2,\ldots,m \qquad (3)$$

Where λ_j,j=1,…,m, are constant not equal to one.

Ordinary differential equations (ODEs)can be considered as a special case of DDEs with λ_j,j=1,…,m, and the theory of DDEs can be considered as a generalization of the theory of ODEs. There are many similarities between the theory of ODEs and that of DDEs, and analytical method for

ODEs have been extended to DDEs when possible. In difference between ODEs and DDEs, such as :

ODEs need to an initial value to determine a unique solution. A system is a finite dimension. DDEs need to an initial function to determine unique solution. A system is a infinite dimension. For example ODE $y'(t) = 3y(t)$ with initial condition $y(t_0)$=a , an example of a simple DDE is $y'(t) = 3y(t-5), t \geq t_0$ with initial function y(t)=φ(t) for $t_0 - 5 \leq t \leq t_0$.

Classification of Delay Differential Equations

Any differential equation in which unknown function appears with various different arguments is called DDEs. The general first DDE with constant form can be presented in the form

$$f(a_0 y'(t), a_1 y'\left(\zeta_j(t)\right), b_0 y(t), b_1 y\left(\zeta_j(t)\right)) = g(t) \qquad (4)$$

Where j=1,…,m , g(t) is a given continuous function and $a_0, a_1, b_0,\ and\ b_1$ are constant. A DDE in (4) is called homogenous if g(t)=0, otherwise is called non-homogenous. There are three types of delay differential equation, which are

1- Retarded delay differential equation
This obtained when ($a_1 = 0\ and\ b_1 \neq 0$) that is

$f(a_0 y'(t), b_0 y(t), b_1 y\left(\zeta_j(t)\right)) = g(t)$ is RDDEs

2- Neutral delay differential equation
This obtained when ($a_1 \neq 0$ and $b_1 = 0$) that is

$f(a_0 y'(t), a_1 y'\left(\zeta_j(t)\right), b_0 y(t)) = g(t)$ is NDDEs

3- Mixed delay differential equation
This obtained when ($a_1 \neq 0$ and $b_1 \neq 0$) that is

$f\left(a_0 y'(t), a_1 y'\left(\zeta_j(t)\right), b_0 y(t), b_1 y\left(\zeta_j(t)\right)\right) = g(t)$ is MDDEs