Chapter One
Numerical Analysis of Ordinary Differential Equation

O.D.E.

I.V.P.	B.V.P.
$y' = f(x,y)$	$y' = f(x,y)$
$y(x_0)=y_0$	$y(x_0)=y_0$ & $y(x_1)=y_1$

Theorem (Existence and uniqueness)

Assume that $f(x, y)$ is continuous function in a region $R=\{(x,y): t_0 \le t \le b,\ c \le y \le d\}$. If f satisfies a Lipschitz condition on R in the variable y and $(x_0,y_0) \in R$, then the initial value problem $y' = f(x,y)$ with $y(x_0)=y_0$ has a unique solution $y=y(x)$ on some subinterival $x_0 \le x \le x_0+\delta$.

The Lipschitz condition is

$|f(x, y_1) - f(x, y_2)| \le K\, |y_1 - y_2|$,
all $(x, y_1), (x, y_2)$ *in* R and $|\partial f(x, y)/\partial y| \le K$

Taylor Series Method

The problem to be solved is a first order ODE

$$\frac{dy(x)}{dx} = f(x,y), \quad y(x_0) = y_0$$

estimates of the solution at different base points $y(x_0+h),\ y(x_0+2h),\ y(x_0+3h),\ ....$ are computed using truncated Taylor series expansions

$$y(x_0+h) \approx y(x_0) + h\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + \frac{h^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{\substack{x=x_0,\\ y=y_0}} + \ldots + \frac{h^n}{n!}\left.\frac{d^ny}{dx^n}\right|_{\substack{x=x_0,\\ y=y_0}}$$

n^{th} order Taylor series method uses n^{th} order Truncated Taylor series expansion

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + ... + \frac{h^n}{n!}\frac{d^ny}{dx^n} + O(h^{n+1})$$

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ..., \frac{d^ny}{dx^n}$$

need to be derived analytically.

Define

$$\left(h\,\frac{\partial}{\partial x}\right)^i f(x,y) = h^i\,\frac{\partial^i f}{\partial x^i}$$

$$\left(h\,\frac{\partial}{\partial x}+k\,\frac{\partial}{\partial y}\right)^0 f(x,y) = f(x,y)$$

$$\left(h\,\frac{\partial}{\partial x}+k\,\frac{\partial}{\partial y}\right)^1 f(x,y) = h\,\frac{\partial\ f(x,y)}{\partial x}+k\,\frac{\partial\ f(x,y)}{\partial y}$$

$$\left(h\,\frac{\partial}{\partial x}+k\,\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2\,\frac{\partial^2\ f(x,y)}{\partial x^2}+2hk\,\frac{\partial^2 f(x,y)}{\partial x\partial y}+k^2\,\frac{\partial^2\ f(x,y)}{\partial y^2}$$

The Taylor series method of order n^{th} has the property that local truncation error (L.T.E.) is of order $O(h^{n+1})$,n can be chosen as large as necessary to make this error as small as desired

First Order Taylor Series Method

$$y(x_0+h) = y(x_0)+h\,\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}}+o(h^2)$$

Notation:

$$x_n = x_0+nh, \qquad y_n = y(x_n),$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=x_i,\\ y=y_i}} = f(x_i,y_i)$$

Euler Method

$$y_{i+1} = y_i + h\ f(x_i,y_i)$$

Euler Method

Problem:

Given the first order ODE $\quad y'(x) = f(x,y)$

with the initial condition $\quad y_0 = y(x_0)$

Determine $\quad y_i = y(x_0+ih) \qquad for\ i = 1,2,...$

Euler Method:

$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\ f(x_i,y_i) \qquad for\ i = 0,1,2,...$$

The local truncation error of Euler method is $O(h^2)$.

Runge-Kutta Methods

— We seek accurate methods to solve ODE that does not require calculating high order derivatives.

— The approach is to a formula involving unknown coefficients then determine these coefficients to match as many terms of the Taylor series expansion.

Second Order Runge Kutta (RK-2)

$$y(x+h) = y(x) + w_1 K_1 + w_2 K_2$$

$$\textit{where} \quad K_1 = h\ f(x, y)$$

$$K_2 = h\ f(x+\alpha h, y+\beta K_1)$$

Problem:

$\textit{Find} \ \alpha, \beta, w_1, w_2$

such that $y(x+h)$ is as accurate as possible.

$\alpha = 1, \beta = 1, w_1 = 1/2, w_2 = 1/2$

The local truncation error of RK2 method is $O(h^3)$.

Third Order Runge Kutta (RK - 3)

$$y(x_i + h) = y(x_i) + \frac{h}{6}\left(K_1 + 4K_2 + K_3\right)$$

$$\textit{where} \quad K_1 = f(x_i, y_i)$$

$$K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h)$$

$$K_3 = f(x_i + \frac{1}{2}h, y_i - K_1 h + 2K_2 h)$$

The local truncation error of RK-3 method is $O(h^4)$.

Fourth Order Runge-Kutta (RK-4)

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$\text{where} \quad k_1 = hf(x_i, y_i)$$
$$k_2 = hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_i + h, y_i + k_3)$$

The local truncation error of RK-4 method is $O(h^5)$.

Problem:

$$\frac{dy}{dx} = 1 + y^2 + x^3, \qquad y(1) = -4$$

$$Use\ RK-2\ to\ find\ y(1.01), y(1.02)$$

Solution :

Step 1:

$$\text{h} = 0.01, x_0 = 1, \quad y_0 = -4, \ f(x,y) = 1 + y^2 + x^3$$

$$x_1 = 1.01, \quad y_1 = y_0 + \frac{h}{2}\left(K_1 + K_2\right) = -3.8254$$

Step 2:

$$K_1 = f(x_1, y_1) = (1 + {y_1}^2 + {x_1}^3) = 16.66$$

$$K_2 = f(x_1 + h, y_1 + K_1 h) = (1 + (y_1 + 0.1666)^2 + (x_1 + .01)^3) = 15.45$$

$$y_2 = y_1 + \frac{h}{2}\left(K_1 + K_2\right) = -3.8254 + \frac{0.01}{2}(16.66 + 15.45) = -3.6648$$

Predictor-Corrector Methods

The methods of Euler,Taylor and Runge-Kutta methods are called single-step methods because they use only the information from one previous point to compute the successive point, i.e.: only the initial point (t_0,y_0) is used to compute (t_1,y_1).

Adams-Bashforth-Moulton Method

The Adams-Bashforth-Moulton predictor-corrector method is multistep method derived from the fundamental theorem of calculus

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f\big(t, y(t)\big)dt \qquad (1)$$

The predictor uses the Lagrange polynomial approximation for f(t,y(t)) based on the points (t_{k-3},f_{k-3}), (t_{k-2},f_{k-2}), (t_{k-1},f_{k-1}) and (t_k,f_k). It is integrating over the interval $[t_k ,t_{k+1}]$. This process produces the Adams-Bashforth predictor:

$$P_{k+1} = y_k + \frac{h}{24}(-9f_{k-3} + 37f_{k-2} - 59f_{k-1} + 55f_k) \qquad (2)$$

The corrector is developed similarly, a second Lagrange polynomial for f(t,y(t)) is constructed, which is based on the points (t_{k-2},f_{k-2}), (t_{k-1},f_{k-1}), (t_k,f_k) and the new point$(t_{k+1},f_{k+1})=(t_{k+1},f(t_{k+1}P_{k+1}))$. This polynomial is then integrated over $[t_k ,t_{k+1}]$ producing the Adams- Moulton corrector:

$$y_{k+1} = y_k + \frac{h}{24}(f_{k-2} - 5f_{k-1} + 19f_k + 9f_{k+1}) \qquad (3)$$

Error estimation and correction

The error terms for the numerical integration formulas used to obtain both the predictor and corrector are of the order $O(h^5)$. The L.T.E. for equations (2) and (3)

$$y(t_{k+1}) - P_{k+1} = \frac{251}{720} y^{(5)}(c_{k+1})h^5 \qquad \text{(L.T.E. for the predictor)} \qquad (4)$$

$$y(t_{k+1}) - y_{k+1} = \frac{-19}{720} y^{(5)}(d_{k+1})h^5 \qquad \text{(L.T.E. for the corrector)} \qquad (5)$$

Suppose that h is small and $y^{(5)}(t)$ is nearly constants over the interval, then the terms involving the fifth derivative in equations (4) and (5) can be eliminated

$$y(t_{k+1}) - y_{k+1} \approx \frac{-19}{720}(y_{k+1} - P_{k+1}) \qquad (6)$$

Formula (6) gives an approximate error estimate based on the two computed values P_{k+1} and y_{k+1} and does not use $y^{(5)}(t)$.

Milne-Simpson Method

Its predictor is based on integration of f(t,y(t)) over the interval $[t_{k-3},t_{k+1}]$:

$$y(t_{k+1}) = y(t_{k-3}) + \int_{t_{k-3}}^{t_{k+1}} f\big(t, y(t)\big)dt \qquad (7)$$

The predictor uses the Lagrange polynomial approximation for f(t,y(t)) based on the points (t_{k-3},f_{k-3}), (t_{k-2},f_{k-2}), (t_{k-1},f_{k-1}) and (t_k,f_k). It is integrating over the interval $[t_{k-3} ,t_{k+1}]$. This process produces the Milne predictor:

$$P_{k+1} = y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_k) \qquad (8)$$

The corrector is developed similarly, The values P_{k+1} can now be used. A second Lagrange polynomial for f(t,y(t)) is constructed, which is based on the points (t_{k-1},f_{k-1}), (t_k,f_k) and the new point$(t_{k+1},f_{k+1})=(t_{k+1},f(t_{k+1}P_{k+1}))$. The polynomial is integrated over $[t_{k-1} ,t_{k+1}]$ and the result is the familiar Simpson's rule :

$$y_{k+1} = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f_{k+1}) \qquad (9)$$

Error estimation and correction

The error terms for the numerical integration formulas used to obtain both the predictor and corrector are of the order $O(h^5)$. The L.T.E. for the formulas (8) and (9) are

$$y(t_{k+1}) - P_{k+1} = \frac{28}{90} y^{(5)}(c_{k+1})h^5 \qquad \text{(L.T.E. for the predictor)} \qquad (10)$$

$$y(t_{k+1}) - y_{k+1} = \frac{-1}{90} y^{(5)}(d_{k+1})h^5 \qquad \text{(L.T.E.for the corrector)} \qquad (11)$$

Suppose that h is small enough so that $y^{(5)}(t)$ is nearly constant over the interval $[t_{k-3},t_{k+1}]$. Then the terms involving the fifth derivative can be eliminated in equations (10) and (11) and the result is

$$y(t_{k+1}) - P_{k+1} \approx \frac{28}{29}(y_{k+1} - P_{k+1}) \qquad (12)$$

Formula (12) gives an approximate error estimate based on the two computed values P_{k+1} and y_{k+1} and does not use $y^{(5)}(t)$. It can be used to improve the predicted value. Under the assumption that the difference between the predicted and corrected values at each step changes slowly, we can substitute P_k and y_k for P_{k+1} and y_{k+1} in (12) and get the following modifier:

$$m_{k+1} = P_{k+1} + \frac{28}{29}(y_k - P_k)$$

This modified value is used in place of P_{k+1} in the correction step, and equation (9) becomes

$$y_{k+1} = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f(t_{k+1}, m_{k+1}))$$

Therefore, the improved (modified) Milne-Simpson method is

$$P_{k+1} = y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_k) \qquad \text{(predictor)}$$

$$m_{k+1} = P_{k+1} + \frac{28}{29}(y_k - P_k)\ , \qquad \text{(modifier)}$$

$$f_{k+1} = f(t_{k+1}, m_{k+1})$$

$$y_{k+1} = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f_{k+1}) \qquad \text{(corrector)}$$

Problem:

Use Adams - Bashforth method to obtain the numerical solution y(0.5) of the I.V.P.

$$y' = 1 + y^2, \qquad y(0) = 0$$

correct to 4 *decimal places, with* $h = 0.1$. *Approximate* $y(0.1), y(0.2)\ y(0.3)$ *using* $RK-4$.

Sol.:

$f(x, y) = 1 + y^2$, $x_0 = 0$, $y_0 = 0$ by using RK-4 we have

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = 0.1003$

$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = 0.1003$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1010$

$$y_1 = y(0.1) = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.1003$$

$y_2 = y(0.2) = 0.2027$

$y_3 = y(0.3) = 0.3093$

According to Adams-Bashforth 4-point method, we have

$$y_4 = y_3 + \frac{h}{24}(-9f_0 + 37f_1 - 59f_2 + 55f_3)$$

$$= 0.3093 + \frac{0.1}{24}\left(-9(1) + 37(1.0101) - 59(1.0411) + 55(1.0957)\right)$$

=0.4227

$$y_5 = y_4 + \frac{h}{24}(-9f_1 + 37f_2 - 59f_3 + 55f_4)$$

$$= 0.4227 + \frac{0.1}{24}\left(-9(1.0101) + 37(1.0411) - 59(1.0957) + 55(1.1787)\right)$$

=0.5462