Sol.:

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2} \quad , \quad p = 0.5$$

j=0 , t_1=0 ⟶ $u(x,0)=4x-4x^2$

$u_{i,1}$=[0 0.64 0.96 0.96 0.64 0]

j=1 , t_2=0.02 ⟶ $u(x_i,t_2) = u_{i,2} = \frac{u_{i-1,1}+u_{i+1,1}}{2}$

$$u_{2,2} = \frac{u_{1,1} + u_{3,1}}{2} = \frac{0 + 0.96}{2} = 0.48$$

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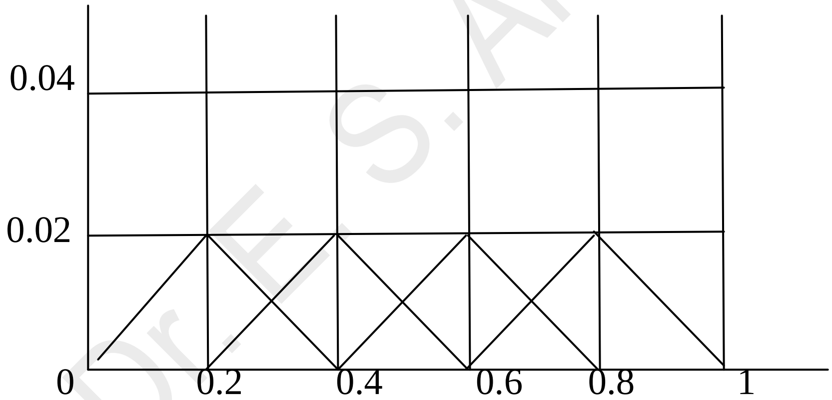
j=2 , t_3=0.04 ⟶ $u(x_i,t_3)$=[0 0.4 0.64 0.64 0.4 0]

and so on .



Local Truncation Error:

We now turn our attention to the accuracy of $u_{i,j}$. let $v_{i,j}$ is the true solution of P.D.E.at the point $(x_i,t_j)=(ih,jk)$

$$\frac{v_{i,j+1}-v_{i,j}}{k}=c\left(\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{h^2}\right)+T_{i,j}$$

Expand in Taylor's series about (x_i,t_j)

$$\frac{v(x,t+k)-v(x,t)}{k}=v_t+\frac{1}{2}kv_{tt}+\frac{1}{6}k^3v_{ttt}+\cdots$$

$$v(x+h,t)-2v(x,t)+v(x-h,t)=2\{\frac{h^2}{2}v_{xx}+\frac{h^4}{24}v_{xxxx}+O(h^6)\}$$

$$T_{i,j}=\frac{v(x,t+k)-v(x,t)}{k}-\frac{c}{h^2}(v(x+h,t)-2v(x,t)+v(x-h,t))$$

$$=v_t+\frac{1}{2}kv_{tt}+\frac{1}{6}k^3v_{ttt}+\cdots-c\{v_{xx}+\frac{h^2}{12}v_{xxxx}+\cdots\}$$

$$=\frac{1}{2}kv_{tt}-c\frac{h^2}{12}v_{xxxx}+\cdots$$

So L.T.E. $=O(k)+O(h^2)$

Numerical Stability:

Consistency is only a necessary but not a sufficient condition for convergence. Roundoff errors incurred during calculations may lead to a blow up of the solution. A scheme is stable if round off errors are not amplified in the calculations. We observe that $\vartheta(t)e^{i\alpha nh}$ is a solution of the difference equation and seek the condition that the term in t=mk does not increase as $m\to\infty$ for stability we require $|\vartheta(t)|\leq 1$ for all α .
Let $u_{i,j}=\vartheta(t)e^{i\alpha nh}$, from the explicit scheme we get

$$\frac{u_{i,j+1}-u_{i,j}}{k}=\frac{\vartheta(t+k)e^{i\alpha nh}-\vartheta(t)e^{i\alpha nh}}{k}=\frac{e^{i\alpha nh}(\vartheta(t+k)-\vartheta(t))}{k}$$

$$\frac{c}{h^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)=\frac{c}{h^2}(\vartheta(t)e^{i\alpha(n+1)h}-2\vartheta(t)e^{i\alpha nh}+\vartheta(t)e^{i\alpha(n-1)h})$$

$$= \frac{c}{h^2}\vartheta(t)e^{i\alpha nh}(e^{i\alpha h} - 2 + e^{-i\alpha h})$$

$$= \frac{c}{h^2}\vartheta(t)e^{i\alpha nh}(2cos\alpha h - 2)$$

$$= \frac{2c}{h^2}\vartheta(t)e^{i\alpha nh}(-2sin^2\left(\frac{\alpha h}{2}\right))$$

Now

$$\frac{\vartheta(t+k) - \vartheta(t)}{k} = \frac{2c}{h^2}\vartheta(t)\left(-2sin^2\left(\frac{\alpha h}{2}\right)\right)$$

$$\vartheta(t+k) - \vartheta(t) = \frac{2ck}{h^2}\vartheta(t)\left(-2sin^2\left(\frac{\alpha h}{2}\right)\right)$$

$$\frac{\vartheta(t+k)}{\vartheta(t)} = 1 + \frac{2ck}{h^2}\left(-2sin^2\left(\frac{\alpha h}{2}\right)\right) = \xi$$

The von Neumann stability condition is

$$\left|\frac{\vartheta(t+k)}{\vartheta(t)}\right| = |\xi| \leq 1$$

$$\Longleftrightarrow \quad |\xi| = \left|1 - 4psin^2\left(\frac{\alpha h}{2}\right)\right| \leq 1 \quad , \quad p = \frac{ck}{h^2}$$

$$\Longleftrightarrow \quad p \leq \frac{1}{2}$$

The explicit scheme is stable if $0 < p \leq \frac{1}{2}$

Convergence:

We have assumed that all the necessary derivatives of (v) exist and from eq.(3) we have

$$v_{i,j+1} = pv_{i+1,j} + (1-2p)v_{i,j} + pv_{i-1,j} + kT_{i,j} \qquad (4)$$

Define $e_{i,j} = v_{i,j} - u_{i,j}$, subtract eq.(3) from eq.(4)

$$e_{i,j+1} = pe_{i+1,j} + (1-2p)e_{i,j} + pe_{i-1,j} + kT_{i,j}$$

The boundary and initial conditions imply

$$e_{0,j} = 0, \qquad e_{n+1,j} = 0\,, \qquad e_{i,0} = 0$$

Define $\epsilon_j = \max_{i=0,n+1}|e_{i,j}|$, then

$|e_{i,j+1}| \le |p||e_{i+1,j}| + |1-2p||e_{i,j}| + |p||e_{i-1,j}| + kT$,

where $T = \max_{i,j}|T_{i,j}|$, if we assume $0 < p \le \frac{1}{2}$ then

$|e_{i,j+1}| \le p\,\epsilon_j + (1-2p)\,\epsilon_j + p\,\epsilon_j + kT$

$= \epsilon_j + kT$ for i=1,2,…,n

This is trivially true for the cases i=0 ,i=n+1

$$\epsilon_{j+1} \le \epsilon_j + kT$$

$\epsilon_0 = 0$ from the I.C.

$\epsilon_1 = kT$, $\epsilon_2 = 2kT$

By induction we can show that

$\epsilon_j \le jkT$ for all $j \ge 0$

Note that $jk = t_j$

$|v_{i,j} - u_{i,j}| \le \epsilon_j \le jkT = t_j T$

We have a convergence if keeping x_i, t_j fixed and letting $i \to \infty$, $j \to \infty$,

$h \to 0$, $k \to 0$ we note that since

$T = O(k) + O(h^2)$, then $T \to 0$ as $h \to 0$, $k \to 0$ and therefore

$$|v_{i,j} - u_{i,j}| \to 0$$

Keeping x_i, t_j fixed and $0 < p \le \frac{1}{2}$.

Crank-Nicholson method

In this method we use the following approximation to u_{xx} at the point (x_i,t_j)

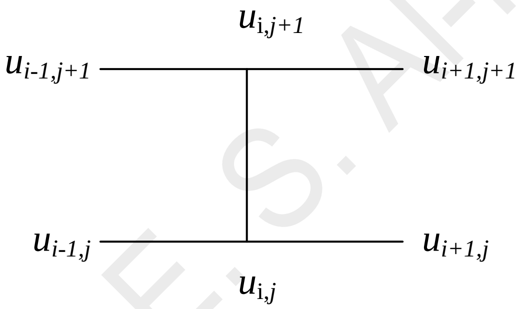
$$u_{xx} = \frac{1}{2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right)$$
$$= \frac{1}{2h^2}\left(\delta_x^2 u_{i,j} + \delta_x^2 u_{i,j+1}\right)$$

Where $\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$, therefore the Crank-Nicolson method for solving eq.(1) is

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{c}{2h^2}\left(\delta_x^2 u_{i,j} + \delta_x^2 u_{i,j+1}\right) \qquad (6)$$

$$-pu_{i-1,j+1} + (1+2p)u_{i,j+1} - pu_{i+1,j+1} = pu_{i-1,j} + (1-2p)u_{i,j} + pu_{i+1,j} \qquad (7)$$

Where $p = \frac{ck}{h^2}$



from eq.(7) we solve for the unknowns along the line t=(j+1)k simultaneously for i=1,2,…,N, eq.(7) represents a system of N equations for the N unknowns $u_{i,j+1}$, i=1,2,…,N we may write this system in the form

$$AU_{j+1} = B$$

Where $U_{j+1} = (u_{1,j+1}, u_{2,j+1}, \ldots, u_{N,j+1})^T$, and

$$A = \begin{bmatrix} (1+2p) & -p & & & \\ -p & (1+2p) & -p & & \\ & -p & (1+2p) & -p & \\ & & & & \\ & & & -p & (1+2p) \end{bmatrix}$$

Is a tridiagonal matrix.

Example: Find the temperature distribution of

$$u_t(x, t) = u_{xx}(x, t) \quad \text{for} \quad 0 \le x \le 1$$

using Crank-Nicolson scheme with the initial and boundary conditions

$u=0$ when $t = 0$ and $0 < x < 1$

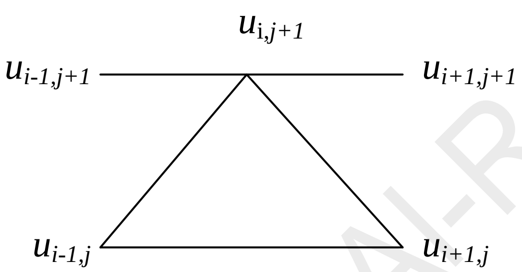
$u=0$ at $x = 0$ and $x = 1$ for $t \ge 0$

take h=0.1 and k=0.01

Sol.: The Crank-Nicolson equation is

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

The computational molecule is given by



j=1 ⟶ $-u_{i-1,2} + 4u_{i,2} - u_{i+1,2} = u_{i-1,1} + u_{i+1,1}$, i=2,3,…,10

i=2 ⟶ $-u_{1,2} + 4u_{2,2} - u_{3,2} = u_{1,1} + u_{3,1} = 1$.

i=3 ⟶ $-u_{2,2} + 4u_{3,2} - u_{4,2} = u_{2,1} + u_{4,1} = 2$

.

.

i=3 ⟶ $-u_{9,2} + 4u_{10,2} - u_{11,2} = u_{9,1} + u_{11,1} = 1$

.

For the first time row, we have the above nine equations corresponding to the nine grid points $u_{2,2}, u_{3,2}, \dots, u_{9,2}$ we have 9×9 system to be solved

$$\begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & & & \\ & & & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{2,2} \\ u_{3,2} \\ u_{4,2} \\ \\ u_{9,2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ \\ 1 \end{bmatrix}$$

This is a tridiagonal system.

Local Truncation Error:

We expand in a Taylor's series about the point (x_i,t_j+1/2k), let $v_{i,j}$ be the true solution of P.D.E. at (x_i,t_j)=(ih,jk). Let $\tau = t_j + \frac{1}{2}k$

$$v\left(\tau + \tfrac{1}{2}k\right) - v\left(\tau - \tfrac{1}{2}k\right) = 2\{\frac{1}{2}kv_t + \frac{(\frac{1}{2}k)^3}{3!}v_{ttt} + \cdots\}$$

$$= kv_t + \frac{k^3}{24}v_{ttt} + \cdots$$

Define $\quad \emptyset(t) = \{v_{i+1} - 2v_i + v_{i-1}\}(t)$

$$= [2\{\frac{h^2}{2}v_{xx} + \frac{h^4}{12}v_{4x} + O(h^6)\}](t)$$

$$\frac{1}{2}\{\emptyset\left(\tau + \tfrac{1}{2}k\right) + \emptyset\left(\tau - \tfrac{1}{2}k\right)\} = \frac{1}{2}[\emptyset(\tau) + \frac{1}{2}k\emptyset'(\tau) + \frac{(\frac{1}{2}k)^2}{2!}\emptyset''(\tau) + \cdots$$

$$+\emptyset(\tau) - \tfrac{1}{2}k\emptyset'(\tau) + \frac{(\frac{1}{2}k)^2}{2!}\emptyset''(\tau) + \cdots]$$

$$= \emptyset(\tau) + \tfrac{1}{8}k^2\emptyset''(\tau) + \cdots$$

$$= h^2v_{xx} + \frac{h^4}{6}v_{4x} + O(h^6) + \frac{k^2}{8}\left\{h^2v_{xxtt} + \frac{h^4}{6}v_{4xtt} + \cdots\right\} + \cdots$$

$$T_{i,j+\frac{1}{2}} = v_t + \tfrac{1}{24}k^2v_{ttt} + \cdots - c\{v_{xx} + \tfrac{h^2}{6}v_{4x} + \tfrac{k^2}{8}h^2v_{xxtt} + \cdots\}$$

$$= -\tfrac{1}{6}ch^2v_{4x} + \tfrac{1}{24}k^2v_{ttt} - \tfrac{1}{8}ch^2k^2v_{xxtt} + \cdots$$

So L.T.E. $\quad T_{i,j+\frac{1}{2}} = O(k^2) + O(h^2)$

Numerical Stability: (H.W.)