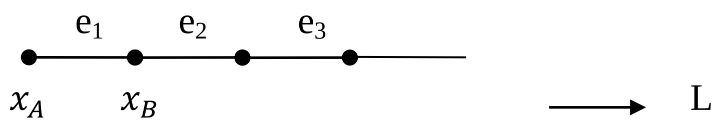
One-dimensional-second order equation

Let $-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f=0$ with u(0)=0 and $a\frac{du}{dx}\Big|_{x=L}=P$

1- Discretization of the domain into a collocation of preselected finite element, let the domain Ω=(0,1)



2- Derivative of element equations

(a)- Variational formulation of the eq. over the element

$$\int_{x_A}^{x_B} N\left(-\frac{d}{dx}\left(a\frac{du}{dx}\right)-f\right)dx=0$$

$$\int_{x_A}^{x_B} \left(a\left(\frac{dN}{dx}\frac{du}{dx}\right)-Nf\right)dx+\left[N\left(-a\frac{du}{dx}\right)\right]\Big|_{x_A}^{x_B}=0$$

$$\int_{x_A}^{x_B} a\left(\frac{dN}{dx}\frac{du}{dx}\right)dx-\left(\int_{x_A}^{x_B} Nf\ dx-N\left(-a\frac{du}{dx}\right)\Big|_{x_A}^{x_B}\right)=0$$

$B(N,u)-l(N)=0$ this is the variational form

(b)- Variational approximation

To find the solution of the variational form with B.C. , we use Galerkin method.

$u^{(e)}(x)=\sum_{j=1}^{n} U_j^{(e)}\,N_j^{(e)}$

$$\sum_{j=1}^{n}\left(\int_{x_A}^{x_B} a\frac{dN_i^{(e)}}{dx}\frac{dN_j^{(e)}}{dx}dx\right)U_j^{(e)} - \int_{x_A}^{x_B} N_i^{(e)} f\ dx - P_1^{(e)}(x_A)N_i^{(e)} - P_2^{(e)}(x_B)N_i^{(e)} = 0$$

$$where\ P_1^{(e)} = \left(-a\frac{du}{dx}\right)\Bigg|_{x_A}\ ,\ \ and\ \ P_2^{(e)} = \left(a\frac{du}{dx}\right)\Bigg|^{x_B}$$

$$\sum_{j=1}^{n} K_{ij}^{(e)}U_j^{(e)} - F_i^{(e)} = 0\ , \qquad i = 1,2,\dots,n$$

$$\left[K^{(e)}\right]U^{(e)} = F^{(e)}$$

Where the coefficient matrix $K_{ij}^{(e)} = B(N_i, N_j)$ is the stiffness matrix and $F_i^{(e)} = l(N_i)$ is the column vector called force vector.

i.e.: $K_{ij}^{(e)} = \int_{x_A}^{x_B} a\frac{dN_i^{(e)}}{dx}\frac{dN_j^{(e)}}{dx}dx$

$$F_i^{(e)} = \int_{x_A}^{x_B} N_i^{(e)} f\ dx - P_1^{(e)}(x_A)N_i^{(e)} - P_2^{(e)}(x_B)N_i^{(e)}$$

After the derivation of the approximation functions $N_i^{(e)}$ for an element ,

$N_1^{(e)}(x) = \frac{x_2 - x}{x_2 - x_1} = \frac{x_B - x}{x_B - x_A}$ and $N_2^{(e)}(x) = \frac{x - x_1}{x_2 - x_1} = \frac{x - x_A}{x_B - x_A}$, $x_A \le x \le x_B$

With the properties:

1- $N_i^e(x_j) = \begin{cases} 0 & if \quad i \ne j \\ 1 & if \quad i = j \end{cases}$

2- $\sum_{i=1}^{2} N_i^e(x_j) = 1$

Then

$$\left[K^{(e)}\right] = \frac{a^{(e)}}{h^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left[F^{(e)}\right] = \frac{f^{(e)}h^{(e)}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} P_1^{(e)} \\ P_2^{(e)} \end{bmatrix}$$

$$\left[K^{(e)}\right]U^{(e)} = F^{(e)}$$

3- Assembly of element equations

Element (1) $\rightarrow$ $$\frac{a_1}{h_1}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{f_1h_1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ 0 \\ 0 \end{bmatrix}$$

Element (2) $\rightarrow$ $$\frac{a_2}{h_2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{f_2h_2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P_1^{(2)} \\ P_2^{(2)} \\ 0 \end{bmatrix}$$

Element (3) $\rightarrow$ $$\frac{a_3}{h_3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{f_3h_3}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ P_1^{(3)} \\ P_2^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{a_1}{h_1} & -\frac{a_1}{h_1} & 0 & 0 \\ -\frac{a_1}{h_1} & \frac{a_1}{h_1}+\frac{a_2}{h_2} & -\frac{a_2}{h_2} & 0 \\ 0 & -\frac{a_2}{h_2} & \frac{a_2}{h_2}+\frac{a_3}{h_3} & -\frac{a_3}{h_3} \\ 0 & 0 & -\frac{a_3}{h_3} & \frac{a_3}{h_3} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} f_1h_1 \\ f_1h_1+f_2h_2 \\ f_2h_2+f_3h_3 \\ f_3h_3 \end{bmatrix} + \begin{bmatrix} P_1^{(1)} \\ P_2^{(1)}+P_1^{(2)} \\ P_2^{(2)}+P_1^{(3)} \\ P_2^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} + F_1^{(3)} \\ F_2^{(3)} \end{bmatrix} \qquad (1)$$

4- Imposition of B.C.

$u_1^{(1)} = u(0) = U_1 = 0$ and $P_2^{(3)} = -\,a \left.\frac{du}{dx}\right|_{x=L} = P$

Then we have

$$\{\Delta\} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad , \quad \{P\} = \begin{bmatrix} -a \left.\frac{du}{dx}\right|_{x=0} = P_1^{(1)} \\ 0 \\ 0 \\ P \end{bmatrix}$$

5- Solution of equations

a is constant , f is constant , $h_1=h_2=h_3=L/3$

the global finite element equations (1) can be partitioned conveniently into the following

$$\begin{bmatrix} [k^{11}] & [k^{12}] \\ [k^{21}] & [k^{22}] \end{bmatrix} \begin{bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \end{bmatrix} = \begin{bmatrix} \{F^1\} \\ \{F^2\} \end{bmatrix} \qquad (2)$$

Where $\{\Delta^1\} = (U_1)$ and $\{\Delta^2\} = (U_2\ ,\ U_3\ ,\ U_4)$

$\{F^1\} = P_1^{(1)}$ and $\{F^2\} = (0\ ,\ 0\ ,\ P)$

Now writing eq.(2) as two matrix equations

$$[k^{11}]\{\Delta^1\} + [k^{12}]\{\Delta^2\} = \{F^1\} \qquad (3)$$

$$[k^{21}]\{\Delta^1\} + [k^{22}]\{\Delta^2\} = \{F^2\} \qquad (4)$$

From eq.(4) we have

$$\{\Delta^2\} = [k^{22}]^{-1}(\{F^2\} - [k^{21}]\{\Delta^1\})$$

$\{F^1\}$ can be computed from eq.(3) we have

$$[k^{11}] = k_{11}^{(1)} \quad , \quad [k^{12}] = [k_{12}^{(1)} \quad 0 \quad 0\,]$$

$$[k^{21}] = \begin{bmatrix} k_{21}^{(1)} \\ 0 \\ 0 \end{bmatrix} \quad , \quad [k^{22}] = \begin{bmatrix} k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix}$$

$$\{F^1\} = F_1^{(1)} \quad , \quad \{F^2\} = \begin{bmatrix} F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} + F_1^{(3)} \\ F_2^{(3)} \end{bmatrix}$$

$$\rightarrow [k^{22}]^{-1} = \frac{L}{3a}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\{\Delta^2\} = \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{fL^2}{18a}\begin{bmatrix} 5 \\ 8 \\ 9 \end{bmatrix} + \frac{PL}{3a}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

And the unknown natural B.C. at x=0

$$P_1^{(1)} = -a\frac{du}{dx}\bigg|_{x=0} = -(fL + P)$$

Example:

$$-\frac{d}{dx}\left((1+x)\frac{du}{dx}\right) - (1+4x) = 0, \quad \text{for } 0 < x < 1$$

with u(0) = u(1) = 0

solution: H.W.