Probability Distribution

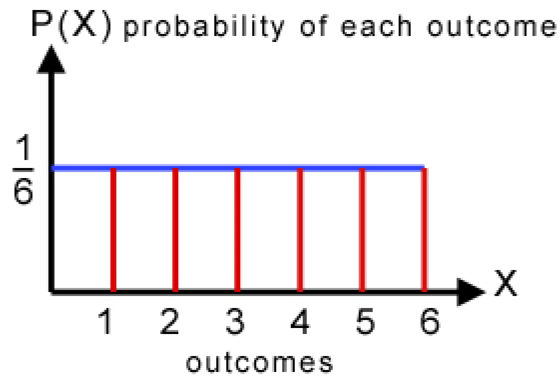
The probability dist. Are divided into two parts which are:

1- Discrete prob. Distribution.
2- Continuous prob. Distribution.

1- Discrete prob. Distribution

1-1- Discrete Uniform Distribution:

The discrete uniform distribution is a symmetric probability distribution with a finite number, and this is mean that any event has the same probability as other events. For example; when we throw a fair die, the possible values are 1,2,3,4,5 and 6, and each time the die is thrown the probability of a given score is 1/6, as the following graph:



From the graph above, some time the uniform discrete called “the rectangular distribution”.

The r.v. X is uniformly distributed with a parameter $"k"$ which is denoted by $Ud(k)$, If and only if the p.m.f. has the following form:

$$f(x) = \begin{cases} \dfrac{1}{k} & X = 1,2,3,....,k \\ 0 & O.W. \end{cases}$$

For a r.v. $x \sim Ud(K)$, We have:

$$E(x) = \frac{K+1}{2} \quad , \quad \text{Var(x)} = \frac{K^2-1}{12}$$

Proof:

$$E(X)=\sum_{X=1}^{K} X \ \text{f(x)}=\frac{1}{K}(1+2+.......+k)$$

$$=\frac{1}{K}.\frac{K(K+1)}{2}=\frac{(K+1)}{2}$$

$$Var(X)=E(X-\mu_X)^2=E(X^2)-\left[E(X)\right]^2$$

$$E(X)^2=\sum_{X=1}^{K} X^2 f(x)=\frac{1}{K}\ .\ \frac{K(K+1)(2K+1)}{6}=\frac{(K+1)(2K+1)}{6}$$

$$Var(X)=\frac{(K+1)(2K+1)}{6}-\left(\frac{(K+1)}{2}\right)^2$$

$$=\frac{K+1}{2}\left[\frac{2K+1}{3}-\frac{K+1}{2}\right]$$

$$=\frac{K+1}{2}\ .\ \frac{K-1}{6}=\frac{K^2-1}{12}$$

This distribution is used indirectly in experiments whose results are characterized by the same opportunity to appear, for example if a simple random sample is drawn from a community, then all samples have the same (equal) opportunity to appear

The moment generating function of Ud.

If $x \sim Ud(K)$, find μ_x^t and $F(x)$, then use μ_x^t to find $E(x)$.

Sol:

$$\mu_x^t=Ee^{tx}=\sum_{X=1}^{K} e^{tx}\ \text{f(x)}=\frac{1}{K}\sum_{X=1}^{K} e^{tx}$$

$$=\frac{1}{K}\left[e^t+e^{2t}+...+e^{Kt}\right]$$

Which the m.g.f of **uniformly discrete** distribution.

The cumulative dist. Of $x \sim Ud(k)$ is:

$$F(x)=P(X\le x)=\sum_{X=1}^{K} f(x)=\frac{1}{K}\sum_{X=1}^{K}(1)=\frac{x}{K}$$

The Expected value is of $x \sim Ud(k)$ by using m.g.f. is:

$$E(x)=\mu_x^{t\,\prime}=\frac{1}{K}\left[e^t+te^{2t}+3e^{3t}+\ldots+Ke^{Kt}\right]$$

$$=\mu_x{}'\Big|_{t=0}=\frac{1}{K}\left[1+2+3+\ldots+K\right]=\frac{K(K+1)}{2K}$$

$$=\frac{K+1}{2}$$

Ex: If $x \sim Ud(4)$, find μ_X , σ_X^2 , F(X) and μ_X^t .

Sol:

$$\mu_X^t=\frac{K+1}{2}=\frac{4+1}{2}=\frac{5}{2}$$

$$\sigma_X^2=\frac{K^2-1}{12}=\frac{16-1}{12}=\frac{15}{12}=\frac{5}{4}$$

$$F(X)=\frac{X}{4} \qquad , \qquad x=1,2,3,4$$

$$\mu_X^t=\frac{1}{4}\sum_{X=1}^{4}e^{tX}=\frac{1}{4}\left[e^t+e^{2t}+e^{3t}+e^{4t}\right]$$

1-2- Bernoulli Distribution:

This distribution is used in experiments whose results include only two cases, which is the state of success that appears with a probability of p value and the state of failure with a probability of 1-p. If the random variable x indicates the battery matching the specification with the probability of success of the attempt p and the probability of failure of 1-p = q, then in this case we use the Bernoulli distribution and it's denoted by $X \sim Ber(p)$, which takes the following form:

$$f(x)=\begin{cases} p^x(1-p)^{1-x} & X=0,1 \\ 0 & OW. \end{cases}$$

Where $p+q=1$ and $q=1-p$ (p is the probability of success and q is the probability of failure).

$p(X=0)=p^0(1-p)^{1-o}=q$ prob. (probability) of failure

$p(X=1)=p^1(1-p)^{1-1}=p$ prob. of success

For a r.v. $X \sim Ber(p)$, we have:

$E(X)=p$, $Var(X)=pq$, $E(X)^m=p$ for all m and $\mu_X^t=pe^t+q$.

Proof:

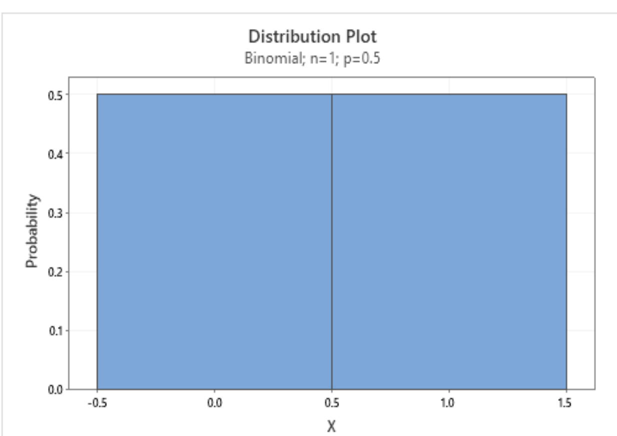
$$E(X)=\sum_{X=0}^{1} X.p^X(1-p)^{1-X}=0.p(X=0)+1.p(X=1)$$
$$=p$$

$$E(X)^2=\sum_{X=0}^{1} X^2.p^X(1-p)^{1-X}=0^2.p(X=0)+1^2.p(X=1)$$
$$=p$$

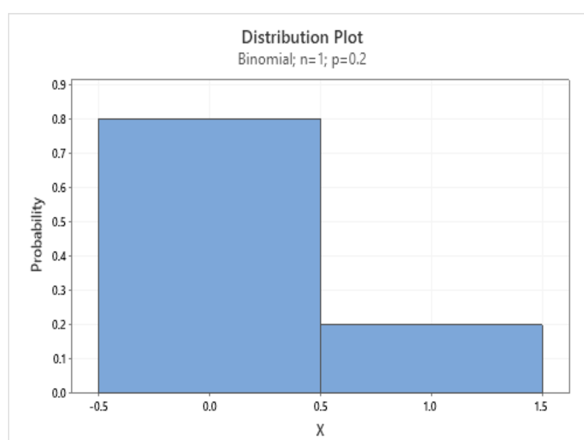
$$E(X)^m=\sum_{X=0}^{1} X^m.p^X(1-p)^{1-X}=0^m.p(X=0)+1^m.p(X=1)$$
$$=p$$

$$\mu_X^t=Ee^{tX}=\sum_{X=0}^{1} e^{tX}.p^X(1-p)^{1-X}=(1-p)+pe^t$$
$$=q+pe^t$$

The graph of this distribution is depending on the value of p, as follows:



when p=0.5



when p=0.2

Ex: Let $X_1, X_2 \sim f(x_1,x_2) = \begin{cases} p^{x_1+x_2}(1-p)^{2-x_1-x_2} & x_1,x_2 = 0,1 \\ 0 & o.w. \end{cases}$

Show that the marginal p.d.f. of X_1 and X_2 is a Bernoulli dist., where the conditional dist. of $X_1 | X_2 = x_2$.

Sol:

From the def. of the conditional dist. we have:

$$f(x_1) = \sum_{x_2=0}^{1} f(x_1,x_2) = p^{x_1}(1-p)^{1-x_1} \sum_{x_2=0}^{1} p^{x_2}(1-p)^{1-x_2}$$

$$= p^{x_1}(1-p)^{1-x_1}.\underbrace{(q+p)}_{=1}$$

$\therefore X_1 \sim Ber(p)$

In the same manner, we get:

$\therefore X_2 \sim Ber(p)$ and $f(x_2) = \sum_{x_1=0}^{1} f(x_1,x_2) = p^{x_2}(1-p)^{1-x_2}$

$$f(X_1 | X_2 = x_2) = \frac{f(x_1,x_2)}{f(x_2)} = \frac{p^{x_1+x_2}(1-p)^{2-x_1-x_2}}{p^{x_2}(1-p)^{1-x_2}}$$

$$= p^{x_1}(1-p)^{1-x_1}$$

Thus X_1 and X_2 are independent.

Ex: If $X \sim Ber(0.3)$, find the mean, variance and m.g.f. of X, then calculate the mean and variance of $Y = 3X + 6$, where Y is a r.v.

Sol:

$$E(X) = \sum_{X=0}^{1} X.p(X)$$

$$= (0).p(X=0) + (1).p(X=1)$$

$$= (0).(0.3)^0(1-0.3)^{1-0} + (1).(0.3)^1(1-0.3)^{1-1}$$

$$= 0 + 0.3 = 0.3$$

$V(X) = E(X)^2 - \left(E(X)\right)^2$

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$$E(X)^2=\sum_{X=0}^{1}X^2.p(X)$$

$$=(o)^2.(0.3)^0(1-0.3)^{1-0}+(1)^2.(0.3)^1(1-0.3)^{1-1}$$
$$=0.3$$

$$\therefore V(X)=E(X)^2-\left(E(X)\right)^2$$

$$=0.3-\left(0.3\right)^2$$

$$=0.21$$

$$\mu_X^t=E(e^{tX})=\sum_{X=0}^{1}e^{tX}.p(X)$$
$$=e^{t(0)}.p(X=0)+e^{t(1)}.p(X=1)$$
$$=e^{t(0)}.(0.3)^0(1-0.3)^{1-0}+e^{t(1)}.(0.3)^1(1-0.3)^{1-1}$$
$$=1+(0.3).e^t$$

$$E(Y)=E(3X+6)=3E(X)+6$$
$$=3(0.3)+6$$
$$=6.9$$

$$V(Y)=V(3X+6)=9V(X)+0$$
$$=9(0.21)$$
$$=1.89$$

H.W.: If $X\sim Ber(0.97)$, Find the mean and variance of $Y=2X+3$.

1-2 Binomial Distribution:

Is the discrete prob. Dist. With two parameters, "n" number of observations (trails) and "p" the prob of success, where all trails are independent. For a single success/ failure experiment is called a Bernoulli trail, it was taken in the last lecture. This dist. Is denoted by $x\sim Bin(n,p)$ iff:

$$f(x)=\begin{cases}\binom{n}{x}p^xq^{n-x} & x=0,1,2,...,n\\ o.w. & 0\end{cases}\qquad 0\le p,q\le 1\qquad q=1-p$$

Where x: No. of success, n-x: No. of failure.
In general:

$\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p+q)^n$, this is called the binomial expansion.

If $x \sim Bin(n,p)$, show that f(x) is a p.d.f.

Sol:

$\sum_{x=0}^{n} f(x) = 1 \rightarrow \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p+q)n$ since p+q=1

$\therefore \sum f(x) = 1$

If $x \sim Bin(n,p)$ then $E(x) = np$, $Var(x) = npq$.

Proof:

$$E(x) = \sum_{all\,(x)} x.f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{(n-x)!} p^x q^{n-x} = \sum_{x=0}^{n} x \frac{n(n-1)!}{(n-x)!} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n-1} \binom{n-1}{x-1} p^{x-1} q^{n-x} = np(p+q)^{n-1}$$

$$= np$$

$$Var(x) = E(x)^2 - [E(x)]^2$$

$$E(x)^2 = E[x(x-1)] + E(x)$$

$$E\left[x(x-1)\right]=\sum_{x=0}^{n} x(x-1)\binom{n}{x} p^{x} q^{n-x}$$

$$=\sum_{x=0}^{n} x(x-1)\frac{n!}{x!(n-x)!} p^{x} q^{n-x}$$

$$=\sum_{x=2}^{n-2} x(x-1)\frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\ (n-x)!} p^{2}\ p^{x-2} q^{n-x}$$

$$=\sum_{x=2}^{n-2} n(n-1)p^{2}\ \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x}$$

$$=\mathrm{n}^{2} p^{2}-np^{2}$$

$$E(x)^{2}=E\left[x(x-1)\right]+E(x)$$

$$=\mathrm{n}^{2} p^{2}-np^{2}+np$$

$$\therefore Var(x)=\mathrm{n}^{2} p^{2}-np^{2}+np-(np)^{2}$$

$$=\mathrm{np}-np^{2}$$

$$=np(1-p)$$

$$=npq$$

The moment generating function of Binomial dist.:

$$\mu_{x}^{t}=Ee^{tx}=\sum_{x=0}^{n} e^{tx}\binom{n}{x} p^{x} q^{n-x}$$

$$=\sum_{x=0}^{n}\binom{n}{x}(pe^{t})^{x} q^{n-x}$$

$$=(\mathrm{p}+\mathrm{qe}^{t})^{n}$$

Ex: Let $x \sim bin(5,\frac{1}{4})$, find:

μ_{x} , $\sigma^{2}{}_{x}$, μ_{x}^{t} and $p(x=1), p(x=0)$ and $\mathrm{p}(\mathrm{x} \geq 1)$

Sol:

$$\mu_x = np = 5\frac{1}{4} = \frac{5}{4}$$

$$\sigma_x^2 = npq = 5\ \frac{1}{4}\ \frac{3}{4} = \frac{15}{16}$$

$$\mu_x^t = (pe^t + q)^n = (\frac{1}{4}e^t + \frac{3}{4})^5$$

$$f(x) = \binom{n}{x} p^x q^{n-x} = C_x^5 (\frac{1}{4})^x (\frac{3}{4})^{n-x}$$

$$p(x=1) = \binom{5}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{5-1} = 0.3955$$

$$\text{p}(x=0) = \binom{5}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{5-0} = 0.2373$$

$$\text{p}(x \geq 1) = 1 - \text{p}(x < 1) = 1 - \text{p}(x = 0)$$

$$= 1 - \binom{5}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{5-0} = 0.7626$$

The Additive property in binomial distribution:

Let $x_1 \sim Bin(n_1, p)$ independent of $x_2 \sim Bin(n_2, p)$, then $Y = X_1 + X_2 \sim Bin(n_1 + n_2, p)$

Proof:

$$\mu_y^t = Ee^{ty} = Ee^{t(X_1 + X_2)} = Ee^{tX_1}\ Ee^{tX_2}$$

$\mu_{X_1}^t = Ee^{tX_1} = (pe^t + q)^{n_1}$ $\qquad$ $\mu_{X_2}^t = Ee^{tX_2} = (pe^t + q)^{n_2}$

$\therefore \mu_Y^t = (pe^e + q)^n$ $\qquad$ where $n = n_1 + n_2$

$\therefore Y \sim Bin(n, p)$

In general, if $Y = \sum_{i=1}^{r} X_i$, such that $X_i \sim Bin(n_i, p)$ independent of $X_j \sim Bin(n_j, p)$, $i, j = 1, 2, ..., r$ then $Y = \sum_{i=1}^{r} X_i \sim Bin(\sum_{j=1}^{r} n_j, p)$

H.W.: If $X_1 \sim Bin(10, \frac{1}{4})$, $X_2 \sim Bin(8, \frac{1}{4})$ and $X_3 \sim Bin(9, p)$, what is the value of p on X_3, then find the prob. Dist. Of $Y = X_1 + X_2 + X_3$, $E(Y)$ and $Var(Y)$.

EX: If $X \sim Bin(n, p)$ derive the dist. of $Y = n - X$.

Sol:

$\mu_Y^t = Ee^{tY} = Ee^{t(n-x)} = e^{nt} \quad Ee^{-tX}$

Since:

$X \sim Bin(n, p) \Rightarrow \mu_X^t = (pe^t + q)^n$

$\therefore \mu_X^{-t} = (pe^{-t} + q)^n$

$\therefore \mu_Y^t = e^{nt} \ (pe^{-t} + q)^n = \left[e^t (pe^{-t} + q)\right]^n$

$= (p + qe^t)^n$

$\therefore Y \sim Bin(n, q)$

$and \quad f(Y) = \binom{n}{Y} q^Y p^{n-Y} \qquad\qquad Y = 0, 1, 2, ..., n$

H.W.: Suppose that $X \sim Bin(3, \frac{1}{3})$, $Y \sim Bin(5, \frac{1}{3})$, Find:

μ_X , $Var(X)$, $E(Y)$, $Var(Y)$, $E(X+Y)$, $Var(X+Y)$, μ^t_{X+Y} and $p(X+Y\geq 1)$

EX: Let $X \sim Bin(n,p)$, Find $E\left(\frac{X}{n}\right)$, $E\left(\frac{(X-np)^2}{n}\right)$, $Cov\left(\frac{X}{n},\frac{n-X}{n}\right)$, $E\left(\frac{X-np}{\sqrt{npq}}\right)$ and $Var\left(\frac{X-np}{\sqrt{npq}}\right)$.

Sol:

$$E\left(\frac{X}{n}\right)=\frac{1}{n}E(X)=\frac{1}{n}np$$

$$=p$$

$$E\left(\frac{(X-np)^2}{n}\right)=\frac{1}{n}E(X-np)^2=-p^2$$

$$Cov\left(\frac{X}{n},\frac{n-X}{n}\right)=E\left(\frac{X}{n},\frac{n-X}{n}\right)$$

$$=E\left(\frac{X}{n}-\frac{1}{n}E(X)\right)\left(\frac{n-X}{n}-\frac{1}{n}E(n-X)\right)$$

$$=E\left(\frac{X}{n}-\frac{1}{n}np\right)\left(\frac{n-X}{n}-\frac{1}{n}(n-np)\right)$$

$$=E\left(\frac{X}{n}-p\right)\left(\frac{n-X}{n}-(1-p)\right)$$

$$=E\left(\frac{X}{n}-p\right)\left(\frac{n-X}{n}-q\right)$$

Complete the solution is **H.W.**

Now:

$$E\left(\frac{X-np}{\sqrt{npq}}\right)=\frac{E(X)-np}{\sqrt{npq}}$$

$$=\frac{np-np}{\sqrt{npq}}=0$$

Last request:

Note that: $Z=\frac{X-np}{\sqrt{npq}}$ represent the standard degree, which is equal to the second moment about original point.

From above:

$Var(Z)=E(Z)^2-\left(E(Z)\right)^2$

Since $E(Z)=0$, then $Var(Z)=E(Z)^2$.

EX: If $X\sim b(n,p)$, find $E(p^x q^{n-x})$, $E(p^x)$, $E(q^{n-x})$ and show that $E(p^x)E(q^{n-x})=(1-pq)^{2n}$

Sol:

$$E(p^x q^{n-x})=\sum_{x=0}^{n}\binom{n}{x} p^x q^{n-x} p^x q^{n-x}=\sum_{x=0}^{n}\binom{n}{x} p^{2x} q^{2(n-x)}$$

$$=\sum_{x=0}^{n}\binom{n}{x}(p^2)^x (q^2)^{n-x}=(p^2+q^2)^n$$

$E(p^x)$ and $E(q^{n-x})$ are H.W.

$$E(p^x)E(q^{n-x}) = (1-pq)^{2n}$$

proof :

$$E(p^x)E(q^{n-x}) = (p^2+q)^n(p+q^2)^n$$
$$= (p^2+(1-p))^n(p+(1-p)^2)^n$$
$$= (p^2+(1-p))^n(p+1-2p+p^2)^n$$
$$= (p^2+(1-p))^n(1-p+p^2)^n$$
$$= (p^2+(1-p))^{2n}$$

EX: If $X \sim b(4,p)$, $Y \sim b(6,p)$ and $p(X \geq 1) = \frac{5}{9}$, find $p(Y > 1)$.

Sol:

$$p(X \geq 1) = \frac{5}{9} \quad \Rightarrow \quad \frac{5}{9} = 1 - p(x=0)$$
$$\Rightarrow p(X=0) = \frac{4}{9}$$

but $p(X=0) = \frac{4!}{0!\ (4-0)!}p^0(1-p)^4 = \frac{0.4}{9}$

$$\Rightarrow (1-p)^2 = \frac{2}{3} \quad \Rightarrow \quad 1-p = +\sqrt{\frac{2}{3}}$$
$$\Rightarrow p = 0.184 \Rightarrow q = 0.814$$

$$\therefore Y \sim b(6, 0.184)$$
$$p(Y > 1) = 1 - p(Y=0) - p(Y=1) = ?$$

H.W.: Let $X \sim b(n,p)$, $\mu_x = 4$ and $\sigma_x^2 = 3$, then find n, p and μ_x^t.

1-3 Poisson Distribution:

The probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event. The r.v. X has a Poisson dist. With parameter λ ,iff:

$$f(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & x=0,1,2,.... \\ o & \text{O.W.} \end{cases}$$

Remark: If $x \sim po(\lambda)$, we have $E(x)=\lambda$ and $Var(x)=\lambda$ and $\mu_x^t = e^{\lambda(e^t-1)}$

Proof:

$$E(x) = \sum_{x=0}^{\infty} x\ f(x) = \sum_{x=0}^{\infty} x\ \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} x\ \frac{e^{-\lambda}\lambda^{x}}{x(x-1)!} = \lambda e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^{x}}{(x-1)!} = \lambda e^{-\lambda}\ e^{\lambda}$$

Where in general:

$$\sum_{y=0}^{\infty}\frac{k^{y}}{y!} = e^{k}$$

$$E(x)^2 = E(x(x-1)) + E(x)$$

$$E(x(x-1)) = \sum_{x=0}^{\infty}(x(x-1))\ f(x) = \sum_{x=0}^{\infty}(x(x-1))\ \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty}(x(x-1))\ \frac{e^{-\lambda}\lambda^{2}\lambda^{x-2}}{x(x-1)(x-2)!} = \lambda^{2}e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} = \lambda^{2}$$

$$E(x)^2 = E(x(x-1)) + E(x)$$

$$= \lambda^{2} + \lambda$$

$$\therefore Var(x) = E(x)^2 - [E(x)]^2$$

$$= \lambda^{2} + \lambda - (\lambda)^{2}$$

$$= \lambda$$

The m.g.f. is:

$$\mu_x^t = Ee^{tx} = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

The additive property of Poisson distribution:

If $x_1, x_2, ..., x_n$ be independent r.v.^{s} and if $x_i \sim po(\lambda_i)$, $i = 1, 2, ..., n$ then $Y = \sum_{i=1}^{n} x_i \sim po(\sum_{i=1}^{n} \lambda_i)$.

Proof:

$$\Phi_Y^{(t)} = E(e^{tY}) = Ee^{t\sum_{i=1}^{n} x_i} = Ee^{tx_1}.Ee^{tx_2}.....Ee^{tx_n}$$

$$= \prod_{i=1}^{n} \mu_{x_i}^t = \prod_{i=1}^{n} e^{\lambda_i (e^t - 1)} = e^{\lambda(e^t - 1)} e^{\lambda(e^t - 1)} .... e^{\lambda(e^t - 1)} = \sum_{i=1}^{n} e^{\lambda_i (e^t - 1)}$$

EX: Let $X \sim Po(\lambda)$ dist. Such that $p(0) = p(1)$, find μ_x, μ_x^t and $p(x > 2)$.

Sol:

$$p(0) = p(1) \Rightarrow \frac{\lambda^0 e^{-\lambda}}{0!} = \frac{\lambda^1 e^{-\lambda}}{1!} \Rightarrow \lambda = 1$$

$$\therefore X \sim Po(1)$$

$$E(X) = \lambda = 1 \text{ , } Var(X) = \lambda = 1$$

$$\mu_X^t = e^{e^t - 1}$$

$$p(X>2)=1-\left[p(X=0)+p(X=1)+p(X=1)\right]$$

$$=1-\left[\frac{e^{-1}}{0!}+\frac{e^{-1}}{1!}+\frac{e^{-1}}{2!}\right]=0.0803$$

$$=1-(0.3678+0.3678+\frac{0.1352}{2})=0.1965$$

EX: If X_1 and X_2 are two r.v.s and independent such that $X_i \sim Po(\lambda_i)$ for $i=1,2$, find the conditional prob. dist. of $X_1|X_1+X_2=n$.

SOL:

$$f\left(X_1|X_1+X_2=n\right)=\frac{f\left(X_1,X_1+X_2=n\right)}{f\left(X_1+X_2=n\right)}$$

Since X_1+X_2 are independent r.v.s then $Z=X_1+X_2 \sim Po(\lambda)$ Where $\lambda=\lambda_1+\lambda_2$ and the p.d.f. of Z is given by:

$$f(Z)=\frac{(\lambda_1+\lambda_2)^Z e^{-(\lambda_1+\lambda_2)}}{Z!} \qquad Z=0,1,2,...$$

$$f(Z=n)=\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}$$

$$f\left(X_1,X_1+X_2=n\right)=f\left(X_1,n-X_1\right)$$

$$=f\left(X_1\right).f\left(n-X_1\right)$$

$$=\frac{(\lambda_1)^{X_1}.e^{-\lambda_1}}{X_1!}.\frac{(\lambda_2)^{n-X_1}.e^{-\lambda_2}}{(n-X_1)!}$$

$$\therefore f\left(X_1|X_1+X_2=n\right)=\frac{n!}{X_1!(n-X_1)!}.\frac{(\lambda_1)^{X_1}(\lambda_2)^{n-X_1}}{(\lambda_1+\lambda_2)^n}.\frac{e^{-(\lambda_1+\lambda_2)}}{e^{-(\lambda_1+\lambda_2)}} \qquad \times \frac{(\lambda_1+\lambda_2)^{X_1}}{(\lambda_1+\lambda_2)^{X_1}}$$

$$f\left(X_1|X_1+X_2=n\right)=\binom{n}{X_1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{X_1}.\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-X_1}$$

Let $p=\frac{\lambda_1}{\lambda_1+\lambda_2}$, $q=1-\frac{\lambda_1}{\lambda_1+\lambda_2}$

$$\therefore f\left(X_1 \mid X_1+X_2=n\right)=\binom{n}{X_1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{X_1}\cdot\left(1-\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-X_1}$$

$$=\binom{n}{X_1}(p)^{X_1}\cdot(q)^{n-X_1}$$

Where $X_1=1,2,\ldots,n$ and $X_1+X_2=n$

1-4 Geometric Distribution:

The r.v.^{s} X has a geometric dist. $G(p)$ with parameter p iff:

$$f(X)=\begin{cases} pq^X & X=0,1,2,\ldots\ldots \\ 0 & \text{O.W.} \end{cases} \quad \text{or} \quad \begin{cases} pq^{X-1} & X=1,2,\ldots\ldots \\ 0 & \text{O.W.} \end{cases}$$

Where the parameter p satisfies $0<p<1$ and $q=1-p$.

Remark:

For a r.v. $X \sim G(p)$, we have:

$$E(X)=\frac{1}{p}, Var(X)=\frac{q}{p^2} \text{ and } \mu_X^t=\frac{pe^t}{1-qe^t}$$

In general, $\sum_{x=0}^{\infty} a^x=\frac{1}{1-a}$ is called inequality geometric.

Prove: The $G(p)$ is the p.m.f. as follows:

$$\sum_{X=1}^{\infty} p(X)=\sum_{X=1}^{\infty} pq^{X-1}=or=\sum_{X=0}^{\infty} pq^X$$

From the inequality above, we have:

$$p\sum_{X=0}^{\infty} q^X=p\frac{1}{1-q}=p\frac{1}{p}=1$$

$\therefore f(X)$ is p.m.f.

$$E(X)=\sum_{X=1}^{\infty}x\ p(x)=\sum_{X=1}^{\infty}x\ pq^{x-1}$$

$$\because \sum_{x=0}^{\infty}xq^{x-1}=\sum_{x=0}^{\infty}\frac{\partial}{\partial q}(q^x)=\frac{\partial}{\partial q}\left(\sum_{x=0}^{\infty}(q^x)\right)$$

$$=\frac{\partial}{\partial q}\left(\frac{1}{1-q}\right)=\frac{1}{(1-q)^2}=\sum_{x=0}^{\infty}xq^{x-1}$$

$$\therefore E(x)=\frac{p}{(1-q)^2}=\frac{1}{p}$$

$$Var(x)=E(x)^2-\left[E(x)\right]^2$$

$$=E(x)^2-\frac{1}{p^2}$$

$$E(x)^2=\sum_{x=1}^{\infty}x^2pq^{x-1}=p\sum_{x=1}^{\infty}x^2q^{x-1}$$

$$=p\sum_{x=1}^{\infty}\left[x(x-1)+x\right]q^{x-1}$$

$$=p\sum_{x=1}^{\infty}x(x-1)q^{x-1}+p\sum_{x=1}^{\infty}xq^{x-1}$$

$$\frac{\partial}{\partial q}q^x=xq^{x-1} \quad \text{and} \quad \frac{\partial^2}{\partial q^2}q^x=\frac{\partial}{\partial q}(xq^{x-1})$$

$$=x(x-1)q^{x-2}$$

$$\therefore E(x)^2=pq\sum_{x=1}^{\infty}x(x-1)q^{x-2}+\frac{1}{p}$$

$$=pq\sum_{x=1}^{\infty}\frac{\partial^2}{\partial q^2}q^x+\frac{1}{p}$$

$$=pq\frac{\partial^2}{\partial q^2}\left(\frac{1}{1-q}\right)+\frac{1}{p}$$

Since:

$$\sum_{x=1}^{\infty} q^x = q + q^2 + q^3 + ... = q(1 + q + q^2 + ...)$$

$$= q\,\frac{1}{1-q}$$

$$\therefore E(x)^2 = \frac{2pq}{p^3} + \frac{1}{p} = \frac{q+1}{p^2}$$

$$\therefore Var(x) = \frac{q+1}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}$$

μ_x^t is **H.W.**

2- Continuous prob. Distribution

1-3- Continuous Uniform Distribution:

The r.v. x has a uniform dist. with two parameters "a" and "b" and it's denoted by $x \sim U(a,b)$, and the p.d.f. of this distribution is:

$$f(X) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & O.W. \end{cases} \quad \text{where } b > a$$

Remark:

a) If $x \sim U(a,b)$ then x is said to be uniformly distributed over the interval (a,b).

b) $U \sim (0.1)$ is called the standard uniform distribution.

c) If the r.v. $x \sim U(a,b)$ then x has the following properties:

$$E(x) = \frac{a+b}{2}$$

$$V(x) = \frac{(b-a)^2}{12}$$

$$\mu_x^t = \frac{e^{bt} - e^{at}}{(b-a).t}$$

$$E(x)^m = \frac{b^{m+1} - a^{m+1}}{(b-a).(m+1)}$$

Note that:

$$E(x-\mu_x)^m = \begin{cases} 0 & for \ \text{"m" is odd} \\ \dfrac{(b-a)^m}{2^m.(m+1)} & for \ \text{"m" is even} \end{cases}$$

Proof of these properties:

$$E(x) = \int_a^b x.f(x).\partial x = \int_a^b x.\frac{1}{b-a}.\partial x$$

$$= \frac{1}{b-a}.\frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{b-a}.\left[\frac{b^2}{2}-\frac{a^2}{2}\right]$$

$$= \frac{b+a}{2}$$

$$V(x) = \int_a^b x^2.f(x).\partial x = \int_a^b x^2.\frac{1}{b-a}.\partial x$$

$$= \frac{1}{b-a}.\frac{x^3}{3}\Big|_a^b$$

$$= \frac{1}{b-a}.\left[\frac{b^3}{3}-\frac{a^3}{3}\right]$$

$$= \frac{(b-a)^3}{3(b-a)}$$

$$= \frac{(b-a).(b^2+ab+a^2)}{3(b-a)}$$

$$= \frac{(b^2+ab+a^2)}{3}$$

$$\therefore V(x) = E(x)^2 - \left(E(x)\right)^2$$

$$= \frac{(b^2+ab+a^2)}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

$$\mu_x^t = E(e^{tx}) = \frac{1}{b-a}\int_a^b e^{tx}.\partial x$$

$$= \frac{1}{t(b-a)}e^{tx}\Big|_a^b$$

$$=\frac{e^{bt}-e^{at}}{t(b-a)} \qquad \text{where} \quad t>0$$

The c.d.f. of x is:

$$F(x)=p(X\le x)=\frac{1}{b-a}\int_a^x f(t).\partial t$$

$$=\frac{1}{b-a}.t\Big|_a^x$$

$$=\frac{x-a}{b-a}$$

The mean deviation of x is:

$$m.d.=E\left|x-E(x)\right|=\frac{1}{b-a}\int_a^b \left|x-E(x)\right|.\partial x$$

$$=\frac{1}{b-a}\int_a^{E(x)} -\left(x-E(x)\right).\partial x+\frac{1}{b-a}\int_{E(x)}^b \left(x-E(x)\right).\partial x$$

$$=\frac{1}{b-a}\left[\int_a^{E(x)} -x.\partial x+\int_a^{E(x)} E(x).\partial x+\frac{1}{b-a}\int_{E(x)}^b x.\partial x-\frac{1}{b-a}\int_{E(x)}^b E(x).\partial x\right]$$

$$=\frac{1}{b-a}\left[\frac{-x^2}{2}\Big|_a^{E(x)}+\frac{E(x)}{2}x\Big|_a^{E(x)}+\frac{x^2}{2}\Big|_{E(x)}^b-\frac{E(x)^2}{2}x\Big|_{E(x)}^b\right]$$

$$=\frac{1}{b-a}\left[\frac{a^2-E(x)^2}{2}+E(x)^2-aE(x)-\frac{b^2-E(x)^2}{2}-b.E(x)+E(x)^2\right]$$

$$=\frac{a^2+b^2-2ab}{4(b-a)}=\frac{b-a}{4}$$

The median of x is:

$$p(X\le x)=p(X\ge x)=\frac{1}{2}$$

Since $F(x)=\dfrac{x-a}{b-a}$

Let $F(x)=\dfrac{1}{2}$

$$\Rightarrow \frac{x-a}{b-a}=\frac{1}{2}\Rightarrow 2(x-a)=b-a$$

$$\therefore \quad x=\frac{a+b}{2}=E(x)$$

Remark:

The uniform continuous distribution does not have a mode.

1-4- Normal distribution:

The normal distribution is one of the most important probability distributions commonly used in statistical theory, since most natural phenomena follow this distribution. The r.v. X has a normal distribution with two parameters μ and σ^2, and it's denoted by $X \sim N(\mu,\sigma^2)$, the p.d.f. of this distribution is:

$$f(X)=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2\sigma^2}(X-\mu)^2} \qquad -\infty < X, \mu < \infty \qquad \sigma^2>0$$

The parameter μ represent the location parameter and the parameter σ^2 represent the shape of parameter.

For a r.v. $X \sim N(\mu,\sigma^2)$ we have:

$$E(X)=\mu \qquad Var(X)=\sigma^2 \qquad \mu_X^t=e^{\mu t+\frac{1}{2}t^2\sigma^2}$$

Remark: if $X \sim N(0,1)$ then we say that X has a **standard normal distribution.**

Ex: Show that the m.g.f. of $X \sim N(\mu,\sigma^2)$ is $\mu_X^t=e^{\mu t+\frac{1}{2}t^2\sigma^2}$.

Sol:

$$\mu_X^t=E(e^{tX})=\int_{-\infty}^{\infty}e^{tX}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2\sigma^2}(X-\mu)^2}\partial X$$

$$=\int_{-\infty}^{\infty}e^{tX}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2\sigma^2}\left(X^2+\mu^2-2\mu X\right)}\partial X$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2\sigma^2}\left(X^2-2\mu X-2\sigma^2 tX\right)-\frac{\mu^2}{2\sigma^2}}\partial X$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2\sigma^2}\left(X^2-2X(\mu+\sigma^2 t)\right)-\frac{\mu^2}{2\sigma^2}}\partial X$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2\sigma^2}\left(X^2-2X(\mu+\sigma^2 t)+(\mu+\sigma^2 t)^2-(\mu+\sigma^2 t)^2\right)-\frac{\mu^2}{2\sigma^2}}\partial X$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2\sigma^2}\left(X-(\mu+\sigma^2 t)\right)^2}.\partial X\ .e^{\frac{1}{2\sigma^2}\left((\mu+\sigma^2 t)^2+\mu^2\right)}$$

$$= e^{\frac{1}{2\sigma^2}\left((\mu^2+(\sigma^2 t)^2+2\mu\sigma^2 t-\mu^2\right)}$$

$$= e^{\frac{1}{2\sigma^2}\left((\sigma^2 t)^2+2\mu\sigma^2 t\right)}$$

$$= e^{\frac{\sigma^2 t}{2}+\mu t}$$

$$E(X) = \frac{\partial \mu_X^t}{\partial t}\Big|_{t=0} = \mu$$

$$E(X)^2 = \frac{\partial^2 \mu_X^t}{\partial t^2}\Big|_{t=0} = \mu^2 + \sigma^2$$

$$\therefore Var(X) = \mu^2 + \sigma^2 + (\mu)^2 = \sigma^2$$

Evaluating the expected and variance of the r.v. X .

$$E(X) = \int_{-\infty}^{\infty} X \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(X-\mu)^2} \partial X$$

Let $Z = \dfrac{X-\mu}{\sigma} \rightarrow X = \sigma Z + \mu \rightarrow \partial X = \sigma . \partial Z \qquad -\infty < Z < \infty$

$$\therefore E(X) = \int_{-\infty}^{\infty} \frac{\sigma Z + \mu}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}(Z)^2} \partial Z$$

$$\frac{\sigma}{\sqrt{2\pi}} e^{\frac{-1}{2}(Z)^2} \Big|_{-\infty}^{\infty} + \mu = \mu$$

$$Var(X) = E(X)^2 - (E(X))^2 = E(X-\mu)^2$$

$$\therefore Var(X) = \int_{-\infty}^{\infty} \frac{E(X-\mu)^2}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(X-\mu)^2} \partial X$$

Let $Z = \dfrac{X-\mu}{\sigma} \rightarrow X - \mu = \sigma Z \rightarrow \partial X = \sigma . \partial Z \qquad -\infty < Z < \infty$

$$\therefore Var(X) = \int_{-\infty}^{\infty} \frac{\sigma^2 Z^2}{\sqrt{2\pi}} e^{\frac{Z^2}{2}} \partial X$$

Let $U = Z \Rightarrow \partial U = \partial Z$

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$$\partial V = Ze^{\frac{-Z^2}{2}}.\partial Z \Rightarrow V = -e^{\frac{-Z^2}{2}}$$

$$\therefore Var(X) = \frac{\sigma^2}{\sqrt{2\pi}}\left[-Ze^{-\frac{Z^2}{2}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\partial Z\right] = \frac{\sigma^2}{\sqrt{2\pi}}\left(\sqrt{2\pi}\right) = \sigma^2$$

The standard normal distribution

If $X \sim N(\mu,\sigma^2)$ then $z = \frac{x-\mu}{\sigma} \sim N(0,1)$

Proof:

$$\mu_z^t = Ee^{tZ} = Ee^{t\left(\frac{x-\mu}{\sigma}\right)}$$

$$= e^{-\frac{\mu t}{\sigma}}.E(e^{\frac{tx}{\sigma}}) = e^{-\frac{\mu t}{\sigma}}.E(e^{t^*x})$$

Where $t^* = \frac{t}{\sigma}$

$$\therefore \mu_z^t = e^{-\frac{\mu t}{\sigma}}.\mu_x^{(t^*)} = e^{-\frac{\mu t}{\sigma}}.e^{\mu t^* + \frac{t^{*2}\sigma^2}{2}}$$

$$= e^{\frac{t^2}{2}} \Rightarrow Z \sim N(0,1)$$

Probability distribution of linear combination:

If $x \sim N(\mu,\sigma^2)$ then $y = ax + b \sim N(a\mu+b, a^2\sigma^2)$.

Proof:

$$\mu_y^t = Ee^{ty} = Ee^{t(ax+b)} = e^{bt}Ee^{t^*x}$$

Where $t^* = at$

$$\therefore \mu_y^t = e^{bt}\mu_x^{t^*} = e^{bt}e^{\mu t^* + \frac{t^*\sigma^2}{2}}$$

$$= e^{(a\mu+b)t + \frac{a^2\sigma^2}{2}}$$

$$\therefore y \sim N(a\mu+b, a^2\sigma^2)$$

In general: if $x_1, x_2, ..., x_n$ are independent r.v.s such that $x_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, ..., n$, then $y = \sum_{i=1}^{n} a_i x_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$.

H.W.: If $x_1, x_2, ..., x_n$ are independent r.v.s, such that $x_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, ..., n$, find the probability distribution of $y = \sum_{i=1}^{n} x_i$.

1-5- Gamma distribution:

This distribution is one of the important distributions in the study of problems in which time is one of its factors, such as those related to the length of operation of units of a particular factory, the study of downtimes for the machines of a particular factory. This distribution is considered one of the important probability distributions in the study of reliability.

The r.v. x has a gamma distribution with parameters α & β and it's denoted by $x \sim G(\alpha, \beta)$, where the p.d.f. is:

$$f(x) = \begin{cases} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x > 0,\ \alpha \text{ and } \beta > 0 \\ 0 & o.w. \end{cases}$$

$\Gamma(\alpha)$ is called the gamma function, this function is defined as:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \partial x \quad \alpha > 0$$

If $\alpha = 1$, then:

$$\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x} . \partial x = \int_0^{\infty} e^{-x} . \partial x = 1$$

$$\Gamma(1) = 1$$

Remark: $\Gamma(\alpha) = (\alpha - 1)!$

Proof:

Let $u = x^{\alpha-1}$ then $\partial u = (\alpha - 1) x^{\alpha-2} \partial x$

And $\partial v = e^{-x}$ then $v = -e^{-x}$

$$\therefore \Gamma(\alpha) = -x^{\alpha-1}e^{-x} \Big|_0^\infty + \int_0^\infty (\alpha-1)x^{\alpha-2}e^{-x}\,.\partial x$$

$$= (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\,.\partial x = (\alpha-1)\Gamma(\alpha-1)$$

$$= (\alpha-1)(\alpha-2)...\Gamma(1)$$

$$= (\alpha-1)!$$

Remark: If $\alpha = 1$ then $\Gamma(1) = 0! = 1$

Ex: how that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sol:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,.\partial x$$

Let $x = \frac{y^2}{2} \Rightarrow \partial x = y\,.\partial y$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{\sqrt{2}}{y} e^{\frac{-y^2}{2}}\,.\partial y = \sqrt{2}\left[\frac{1}{2}\int e^{\frac{-y^2}{2}}\,.\partial y\right]$$

$$= \sqrt{2}\left[\frac{1}{2}\sqrt{2\pi}\right]$$

$$= \sqrt{\pi}$$

Remark: If $x \sim G(\alpha, \beta)$, then:

1) If $\alpha = 1$, the gamma reduces to the exponential dist. with parameter β.
2) If $\alpha = \frac{k}{2}$ and $\beta = \frac{1}{2}$, then the gamma dist. will be recognized as the chi-square dist. with parameter k, which is known as the degrees of freedom.

Remark: If $x \sim G(\alpha,\beta)$, then:

$$E(x)=\frac{\alpha}{\beta},\ Var(x)=\frac{\alpha}{\beta^2} \text{ and } \mu_x^t=\left(\frac{\beta}{\beta-t}\right)^{\alpha}.$$

Proof:

If $\therefore E(x)=\int_0^\infty x\,\frac{\beta^\alpha}{\Gamma\alpha}x^{\alpha-1}e^{-\beta x}.\partial x=\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty x^{\alpha}e^{-\beta x}.\partial x$

By multiply the right side by $\frac{\beta\Gamma\alpha+1}{\beta\Gamma\alpha+1}$, we have:

$$E(x)=\frac{\beta^\alpha}{\Gamma\alpha}.\frac{\beta\Gamma\alpha+1}{\beta\Gamma\alpha+1}\int_0^\infty x^{\alpha}e^{-\beta x}.\partial x=\frac{\Gamma\alpha+1}{\beta\Gamma\alpha}\int_0^\infty\frac{\beta^{\alpha+1}}{\Gamma\alpha+1}x^{\alpha}e^{-\beta x}.\partial x$$

$$=\frac{\Gamma\alpha+1}{\beta\Gamma\alpha}=\frac{\alpha\Gamma\alpha}{\beta\Gamma\alpha}=\frac{\alpha}{\beta}$$

$$E(x)^2=\int_0^\infty x^2\,\frac{\beta^\alpha}{\Gamma\alpha}x^{\alpha-1}e^{-\beta x}.\partial x$$

By multiply the right side by $\frac{\beta^2\Gamma\alpha+2}{\beta^2\Gamma\alpha+2}$, we have:

$$\therefore E(x)^2=\frac{\beta^2\Gamma\alpha+2}{\beta^2\Gamma\alpha+2}\int_0^\infty\frac{\beta^\alpha}{\Gamma\alpha}x^{\alpha+1}e^{-\beta x}.\partial x=\frac{\Gamma\alpha+2}{\Gamma\alpha\,\beta^2}\int_0^\infty\frac{\beta^{\alpha+2}}{\Gamma\alpha+2}x^{\alpha+1}e^{-\beta x}.\partial x$$

$$=\frac{\Gamma\alpha+2}{\Gamma\alpha\,\beta^2}=\frac{\alpha(\alpha+1)\Gamma\alpha}{\Gamma\alpha\,\beta^2}=\frac{\alpha(\alpha+1)}{\beta^2}$$

$$\therefore Var(x)=\frac{\alpha(\alpha+1)}{\beta^2}-\left(\frac{\alpha}{\beta}\right)^2=\frac{\alpha}{\beta^2}$$

The moment generating function:

$$\mu_x^t=\int_0^\infty e^{tx}\,\frac{\beta^\alpha}{\Gamma\alpha}x^{\alpha-1}e^{-\beta x}.\partial x=\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty x^{\alpha-1}e^{-x(\beta-t)}.\partial x$$

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By multiply the right side by $\frac{(\beta-t)^{\alpha}}{(\beta-t)^{\alpha}}$, we get:

$$\mu_x^t = \frac{\beta^{\alpha}}{\Gamma\alpha}\frac{(\beta-t)^{\alpha}}{(\beta-t)^{\alpha}}\int_0^{\infty} x^{\alpha-1}e^{-x(\beta-t)}.\partial x$$

$$= \frac{\beta^{\alpha}}{(\beta-t)^{\alpha}}\int_0^{\infty}\frac{(\beta-t)^{\alpha}}{\Gamma\alpha}x^{\alpha-1}e^{-x(\beta-t)}.\partial x = \frac{\beta^{\alpha}}{(\beta-t)^{\alpha}}$$

$$= \left(\frac{1}{1-\frac{t}{\beta}}\right)^{\alpha} = \left(1-\frac{t}{\beta}\right)^{\alpha}$$

Remark: there is another formula for gamma dist. which is defined as:

$$f(X;\alpha,\beta) = \frac{1}{\Gamma\alpha\,\beta^{\alpha}}X^{\alpha-1}e^{\frac{-X}{\beta}} \quad X,\alpha,\beta>0$$

In this case $E(X)=\alpha\beta$, $Var(X)=\alpha\beta^2$ and $\mu_X^t=(1-\beta t)^{-\alpha}$ where $X \sim G(\alpha,\frac{1}{\beta})$.

Remark: If $X \sim G(\alpha,1)$, then the p.d.f. is:

$$f(X) = \frac{1}{\Gamma\alpha}X^{\alpha-1}e^{-X}$$

Ex: If $X_1,X_2,...,X_n$ be independent r.v.^{s} such that $X_i \sim G(\alpha_i,\beta)$ for $i=1,2,...,n$, then $Y=\sum_{i=1}^{n}X_i \sim G\left(\sum_{i=1}^{n}\alpha_i,\beta\right)$.

Proof:

$$\mu_Y^t = Ee^{tY} = Ee^{t\sum_{i=1}^{n}X_i} = \prod_{i=1}^{n}Ee^{tX_i}$$

$$= \prod_{i=1}^{n}\mu_{X_i}^t = \prod_{i=1}^{n}\left(1-\frac{t}{\beta}\right)^{-\alpha_i} = \left(1-\frac{t}{\beta}\right)^{-\sum_{i=1}^{n}\alpha_i}$$

This is the m.g.f. of $G(\sum_{i=1}^{n}\alpha_i,\beta)$

$$\therefore Y \sim G(\sum_{i=1}^{n}\alpha_i,\beta)$$

$$f(Y)=\frac{\beta^{\sum_{i=1}^{n}\alpha_i}}{\Gamma\sum_{i=1}^{n}\alpha_i}Y^{\sum_{i=1}^{n}\alpha_i-1}e^{-\beta Y}$$

Ex: If $X_1,X_2,...,X_n$ be independent r.v.^{s} such that $X_i \sim Exp(\theta)$ for $i=1,2,...,n$, show that $Y=\sum_{i=1}^{n}X_i \sim G(n,\theta)$.

Proof:

$$\mu_Y^t = Ee^{tY} = Ee^{t\sum_{i=1}^{n}X_i} = \prod_{i=1}^{n}Ee^{tX_i} = \prod_{i=1}^{n}\mu_{X_i}^t$$

$$\Rightarrow \prod_{i=1}^{n}\frac{\theta}{\theta-t}=\left(\frac{\theta}{\theta-t}\right)^n=\left(1-\frac{t}{\theta}\right)^{-n}$$

Which is the m.g.f of Gamma dist., then $Y \sim G(n,\theta)$.

Ex: Let $X \sim G(2,2)$, find $P(X \le 3)$.

Sol:

$$f(X)=\frac{2^2}{\Gamma 2}X^{2-1}e^{-2X}=4Xe^{-2X}$$

$$\therefore p(X\le 3)=\int_0^3 4X.e^{-2X}.\partial X$$

Let $u=X \Rightarrow \partial u=\partial X$ and $\partial v=e^{-2X}\partial X \Rightarrow v=\frac{-1}{2}e^{-2X}$

$$\therefore p(X\le 3)=4\left[\frac{-X}{2}.e^{-2X}\Big|_0^3+\frac{1}{2}\int_0^3 e^{-2X}.\partial X\right]$$

$$=4\left[\frac{-3}{2}.e^{-6}-\frac{1}{4}(e^{-6}-1)\right]$$

$$=-6e^{-6}-e^{-6}+1=1-7e^{-6}=\frac{e^{6}-7}{e^{6}}$$

Ex: Let $X \sim G(\alpha,\beta)$, find the m.g.f. of $Y=\ln(X)$.

Sol:

By using the m.g.f. method, we have:

$$\mu_Y^t = Ee^{tY} = Ee^{t.\ln(X)} = Ee^{\ln(X)^t} = E(X)^t$$

$$=\frac{\beta^{\alpha}}{\Gamma\alpha}\int_0^{\infty} X^{t} X^{\alpha-1} e^{-\beta X}.\partial X$$

$$=\frac{\beta^{\alpha}}{\Gamma\alpha}\int_0^{\infty} X^{t+\alpha-1} e^{-\beta X}.\partial X$$ by multiple $\frac{\beta^{t+\alpha}\Gamma(t+\alpha)}{\beta^{t+\alpha}\Gamma(t+\alpha)}$, we have:

$$=\frac{\beta^{\alpha}.\Gamma(t+\alpha)}{\beta^{t+\alpha}.\Gamma\alpha}\int_0^{\infty}\frac{\beta^{t+\alpha}}{\Gamma(t+\alpha)}.X^{t+\alpha-1}e^{-\beta X}.\partial X$$

$$=\frac{\beta^{\alpha}.\Gamma(t+\alpha)}{\beta^{t+\alpha}.\Gamma\alpha}=\frac{\Gamma(t+\alpha)}{\beta^{t}.\Gamma\alpha}$$

Ex: Let $X \sim G(\alpha,\beta)$, find $E(e^{X})$.

Sol:

$$E(e^{X})=\frac{\beta^{\alpha}}{\Gamma\alpha}\int_0^{\infty} e^{X}.X^{\alpha-1}e^{-\beta X}.\partial X = \frac{\beta^{\alpha}}{\Gamma\alpha}\int_0^{\infty} X^{\alpha-1}e^{-(\beta-1)X}.\partial X$$ By multiple

$\frac{(\beta-1)^{\alpha}}{(\beta-1)^{\alpha}}$, we have:

$$E(e^{X})=\frac{\beta^{\alpha}}{(\beta-1)^{\alpha}}\int_0^{\infty}\frac{(\beta-1)^{\alpha}}{\Gamma\alpha}X^{\alpha-1}e^{-(\beta-1)X}.\partial X=\frac{\beta^{\alpha}}{(\beta-1)^{\alpha}}=\left(\frac{\beta}{\beta-1}\right)^{\alpha}$$

1-6- **Exponential distribution**:

The r.v. X has an exponential distribution with parameter λ, and it's denoted by $X \sim Exp(\lambda)$, where the p.d.f. is:

$$f(x) = \begin{cases} \lambda e^{-\lambda X} & X > 0 \\ 0 & O.W. \end{cases}$$

Remark:

For a r.v. $X \sim Exp(\lambda)$, we have:

$$E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2} \qquad \mu_X^t = \frac{\lambda}{\lambda - t}$$

Proof:

$$E(X) = \int_0^\infty X.f(X).\partial X = \int_0^\infty X.\lambda e^{-\lambda X}.\partial X$$

Using integration by parts, we have:

$$u = X \rightarrow \partial u = \partial X$$

$$\partial v = \int \lambda e^{-\lambda X}.\partial v \rightarrow v = -e^{-\lambda X}$$

$$\therefore E(X) = -Xe^{-\lambda X} \Big|_0^\infty - \int_0^\infty -e^{-\lambda X}.\partial X$$

$$= 0 + \frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty$$

$$= \frac{-1}{\lambda} e^{-\infty} + \frac{1}{\lambda} e^0 = \frac{1}{\lambda}$$

$$E(X)^2 = \int_0^\infty X^2.f(X).\partial X = \int_0^\infty X^2.\lambda e^{-\lambda X}.\partial X$$

$$u = X^2 \rightarrow \partial u = 2X.\partial X \qquad \partial v = \int \lambda e^{-\lambda X}.\partial v \rightarrow v = -e^{-\lambda X}$$

$$\therefore E(X)^2 = -X^2 e^{-\lambda X} \Big|_0^\infty - \int_0^\infty -2Xe^{-\lambda X}.\partial X$$

$$= 0 + \int_0^\infty 2Xe^{-\lambda X}.\partial X = 2\int_0^\infty Xe^{-\lambda X}.\partial X$$

$$u = X \to \partial u = \partial X \qquad \partial v = \int e^{-\lambda X} . \partial v \to v = \frac{-1}{\lambda} e^{-\lambda X}$$

$$\therefore E(X)^2 = 2\left(X \frac{-1}{\lambda} e^{-\lambda X} - \int_0^\infty \frac{-1}{\lambda} e^{-\lambda X} . \partial X \right)$$

$$= 2\left(0 + \int_0^\infty \frac{-1}{\lambda} e^{-\lambda X} . \partial X \right)$$

$$= 2 \frac{-1}{\lambda^2} e^{-\lambda X} \Big|_0^\infty = \frac{2}{\lambda^2}$$

$$\therefore Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{1}{\lambda^2}$$

H.W.: find $Var(X)$ where $X \sim Exp(\lambda)$.

$$\mu_X^t = Ee^{tX} = \int_0^\infty e^{tX} . \lambda e^{-\lambda X} . \partial X$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)X} . \partial X$$

$$= \frac{-\lambda}{\lambda - t} e^{-(\lambda - t)X} \Big|_0^\infty = \frac{-\lambda}{\lambda - t} (e^{-\infty} - e^0)$$

$$= \frac{\lambda}{\lambda - t} \qquad t < \lambda$$

Remark:

There is another shape for the p.d.f. of the exponential distribution which is:

$$f(X; \lambda) = \frac{1}{\lambda} e^{\frac{-X}{\lambda}} \qquad X > 0$$

In this case $E(X) = \lambda$ and $Var(X) = \lambda^2$

The distribution function is:

$$F(X) = p(X \le x) = \int_0^x \lambda e^{-\lambda X} . \partial X = -e^{-\lambda X} \Big|_0^x = 1 - e^{-\lambda x}$$

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Note that:

$$\frac{\partial \mu_x^t}{\partial t}\Big|_{t=0} = E(x) = \frac{1}{\lambda}$$

$$\frac{\partial^2 \mu_x^t}{\partial t^2}\Big|_{t=0} = E(x)^2 = \frac{2}{\lambda^2}$$

In the exponential dist. $E(X) > Var(X)$.

Ex: If $X \sim Exp(\lambda)$, find the median of this dist.

Sol:

$$F(X) = \frac{1}{2}$$

$$\therefore 1 - e^{-\lambda x} = \frac{1}{2} \Rightarrow -e^{-\lambda X} = \frac{-1}{2} \Rightarrow -\lambda X = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow X = \frac{\ln(2)}{\lambda}$$

H.W. If $X \sim Exp(4)$, find $E(X), Var(X), \mu_X^t$ and $F(X)$.

1-7- **Chi-square distribution**:

This distribution is a special case of Gamma distribution when $\alpha = \frac{n}{2}$ and $\beta = \frac{1}{2}$, then the p.d.f of this distribution of the r.v. X is:

$$f(\chi^2, n) = \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}} (\chi^2)^{\frac{n}{2}-1} e^{\frac{-1}{2}\chi^2} \text{ where } X > 0$$

Where n is number of degrees of freedom.

$E(X) = n$, $Var(X) = 2n$ and $\mu_X^t = (1-2t)^{\frac{-n}{2}}$.

Proof:

$$E(X)=\int_0^\infty X\,\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}}(X)^{\frac{n}{2}-1}e^{\frac{-1}{2}X}\,\partial X$$

Let $Y=\dfrac{X}{2}\Rightarrow X=2Y$

$\therefore \partial X = w\,\partial Y$

$$\therefore E(X)=\int_0^\infty 2Y\,\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}}(2Y)^{\frac{n}{2}-1}e^{\frac{-1}{2}2Y}\,2\partial Y=\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}}.2^{\frac{n}{2}}.\int_0^\infty 2.(Y)^{\frac{n}{2}}e^{\frac{-1}{2}Y}\,\partial Y$$

$$=\frac{2}{\Gamma\frac{n}{2}}.\Gamma\left(\frac{n}{2}+1\right)$$

$$=\frac{2}{\Gamma\frac{n}{2}}.\Gamma\frac{n}{2}.\frac{n}{2}=n$$

$$E(X)^2=\int_0^\infty X^{\,2}\,\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}}(X)^{\frac{n}{2}-1}e^{\frac{-1}{2}X}\,\partial X=\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}}\int_0^\infty (X)^{\frac{n}{2}+2-1}e^{\frac{-1}{2}X}\,\partial X$$

$$=\frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}}.\frac{\Gamma\left(\frac{n}{2}+2\right)}{\left(\frac{1}{2}\right)^{\frac{n}{2}+2}}$$

$$=\frac{4.\left(\frac{n}{2}+1\right).\frac{n}{2}.\Gamma\frac{n}{2}}{\Gamma\frac{n}{2}}=n^2+2n$$

$$\therefore Var(X)=n^2+2n-(n)^2=2n$$

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$$\mu_X^t = Ee^{tX} = \int_0^\infty e^{tX} \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}} (X)^{\frac{n}{2}-1} e^{\frac{-1}{2}X} \partial X = \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}} . \int_0^\infty (X)^{\frac{n}{2}-1} e^{-(\frac{1}{2}-t)X} \partial X$$

$$= \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\frac{n}{2}} . \frac{\Gamma\frac{n}{2}}{(\frac{1}{2}-t)^{\frac{n}{2}}} = (1-2t)^{\frac{-n}{2}}$$

Ex: Prove that if $X \sim N(\mu,\sigma^2)$, then $Z^2 = \left(\frac{X-\mu}{\sigma^2}\right) \sim \chi^2_{(1)}$.

Sol:

By using m.g.f. method, we have:

$$\mu_Z^t = Ee^{tZ} = \int_{-\infty}^{\infty} e^{tZ^2} .f(Z).\partial Z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tZ^2} e^{\frac{-Z^2}{2}} \partial Z$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(1-2t)Z^2} \partial Z$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\sqrt{(1-2t)}Z\right)^2} \partial Z$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{Z-0}{1/\sqrt{(1-2t)}}\right)^2} \partial Z$$

$$= \frac{1}{\sqrt{2\pi}} . \frac{\sqrt{2\pi}}{\sqrt{1-2t}} = (1-2t)^{\frac{-1}{2}}$$

Which is the m.g.f. of Chi-square with (1) degrees of freedom.

Ex: Let $X_1, X_2, ..., X_r$ be independent r.v.^{s}, such that $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2, ..., r$, then $u = \sum_{i=1}^{r} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi^2_r$.

Sol:

$$u = \sum_{i=1}^{r} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 = \sum_{i=1}^{r} Z_i^2$$

By using m.g.f., we have:

$$\mu_u^t = Ee^{tu} = Ee^{t\sum_{i=1}^{r} Z_i^2} = Ee^{t(Z_1^2 + Z_2^2 + ... + Z_r^2)}$$

$$= \prod_{i=1}^{r} E e^{tZ_i^2}$$

$$f(Z_i) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2} Z_i^2}$$

$$\therefore E e^{tZ_i^2} = \int_{-\infty}^{\infty} e^{tZ_i^2} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2} Z_i^2} \partial Z_i = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tZ_i^2 - \frac{1}{2} Z_i^2} \partial Z_i$$

$$= (1-2t)^{\frac{-1}{2}}$$

$$\therefore \mu_u^t = \prod_{i=1}^{r} (1-2t)^{\frac{-1}{2}} = (1-2t)^{\frac{-r}{2}}$$

Which is the m.g.f. of Chi-square with r degree of freedom, i.e.
$u \sim \chi^2_{(r)}$

Ex: Let $X_1, X_2, ..., X_r$ be independent r.v.s, such that $X_i \sim \chi^2_{(r)}$, $i = 1, 2, ..., r$, then $Y = \sum_{i=1}^{n} X_i \sim \chi^2_{\sum_{i=1}^{n} r_i}$.

Proof:

By using m.g.f. method:

$$\mu_Y^t = E e^{tY} = E e^{t \sum_{i=1}^{n} X_i}$$

$$= \prod_{i=1}^{n} E e^{tX_i} = \prod_{i=1}^{n} \mu_{X_i}^t$$

Since $X_i \sim \chi^2_{(r)}$ then:

$$\mu_{X_i}^t = (1-2t)^{\frac{-r_i}{2}}$$

$$\therefore \mu_Y^t = (1-2t)^{\frac{-\sum_{i=1}^{n} r_i}{2}}$$

Ex: If $X_1 \sim \chi^2_{(3)}$, $X_2 \sim \chi^2_{(7)}$ and $X_3 \sim \chi^2_{(8)}$, then $Y = X_1 + X_2 + X_3 \sim \chi^2_{18}$.

$$\therefore f(Y) = \frac{\left(\frac{1}{2}\right)^9}{\Gamma 9} Y^{9-1} e^{\frac{-Y}{2}}$$

Ex: Let $\overline{X}$ denote the sample mean of a r.s. of size n drown from $N(\mu,\sigma^2)$, then $Y = \left(\frac{\sqrt{n}(\overline{X}-\mu)}{\sigma}\right)^2 \sim \chi^2_{(1)}$.

Sol:

We know that $X \sim N(\mu,\sigma^2)$ then $\overline{X} \sim N(\mu,\frac{\sigma^2}{n})$

$$\therefore Z = \frac{\sqrt{n}(\overline{X}-\mu)}{\sigma} \sim N(0,1)$$

$$\therefore Y = Z^2 = \frac{n(\overline{X}-\mu)^2}{\sigma^2} \sim \chi^2_{(1)}$$

Ex: Let $S^2 = \frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}$ be the sample variance of a r.s. of size n, drown from $N(\mu,\sigma^2)$ then $u = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$.

Proof:

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}\left[(X_i-\overline{X})+(\overline{X}-\mu)\right]^2$$

$$= \sum_{i=1}^{n}(X_i-\overline{X})^2 + n(\overline{X}-\mu)^2$$

$$\because \sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2} = \sum_{i=1}^{n}\frac{(X_i-\overline{X})^2}{\sigma^2} + \frac{n(\overline{X}-\mu)^2}{\sigma^2}$$

$$= \frac{(n-1)S^2}{\sigma^2} + \frac{n(\overline{X}-\mu)^2}{\sigma^2}$$

$$\therefore \chi^2_{(n)} = \frac{(n-1)S^2}{\sigma^2} + \chi^2_{(1)}$$

$$\therefore \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$