Chapter one

Set theory

1.1 - Basic set theory قواعد نظرية المعلومات

Definitions:

1- ***Set*** **:** A set is collection of objects, the objects are called element (Events) of the set .

المجموعة : هي عبارة عن أشياء مجمعة ، وهذه الأشياء تسمى (عناصر او حوادث) في مجموعة .

A set is usually denoted by capital letter e.g. A,B,C,… ; If X is an event belonging to the set ***A*** , we shall write $x \in \boldsymbol{A}$ if x is not an event or element of ***A*** , we shall $x \notin \boldsymbol{A}$.

Exp) ***A***={0,1,2,3,4} and ***B***={ X : X < 8 } where (X : positive integer عدد صحيح موجب)

∴ **B** = { 1,2,3,4,5,6,7 }

2- ***Sample space*** ($\boldsymbol{S}$, $\boldsymbol{\Omega}$) **:** The all possible outcomes of a random experiment are called the sample space .

Exp) The experiment of tossing a coin is :

$$\boldsymbol{S} = \{ \boldsymbol{H} , \boldsymbol{T} \}$$

3- ***Random Experiment :*** All outcomes of the experiment known . وهي كل المخرجات من التجربة المعلومة

4- ***Events*** : An event is any collection of possible outcomes of an experiment that is any subset of $\boldsymbol{S}$

الحدث : أي ترتيب او مجموعة محتمل ظهوره في التجربة وهو مجموعة جزئية من فضاء العينة

Exp) $\boldsymbol{S} = \{ \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6} \}$

$\boldsymbol{X} = \{ \mathbf{3} \}$

$\boldsymbol{Y} = \{ \mathbf{2}, \mathbf{3} \}$

$\boldsymbol{Z} = \{ \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6} \} = \boldsymbol{S}$

5- ***Subset*** : The set A is said to be a subset of S **if** every element of A is also an element of S .

يقال ان A مجموعة جزئية من فضاء الحالة او العينة ، اذا كانت كل العناصر الموجودة في A أيضا موجودة في فضاء العينة

i.e. A is contained in S ; that is mean :

$x \in (A \subset S)$

$x \in A \ \ and \ \ x \in S$

6-***Empty set*** : A set is said to be an empty set or null set if it has no element , and is denoted by $\emptyset$;

$\emptyset \subset S$

7-***Equal set***: Two set A and B are said to be equal if every elements in A are contained in B also .

$A \subset B$ and $B \subset A$

Then **A = B**

8-***Universal set***: All set under investigation are assumed to be subset of the space , we shall call it the space and denote it by S or Ω **.**

9-***The complement***: The complement of the set A with respect to the S , denoted by $(\overline{A}, A^c, A')$ it is the set of all the element that are in S but not in A .

That is mean : $A^c = \{x : x \in s \ \ and \ \ x \notin A\}$.

10-***set difference***: Let A and B be any two subset of S , the set of all points in A that are not in B , will denoted by $(A/B \ \ or \ \ A - B)$

That is mean :- $A/B = \{x : x \in A \ \ and \ \ x \notin B\}$

$A^c = S/A$

$A^c = \{x : x \in S \ \ and \ \ x \notin A\}$

- If A is a subset of B then $A \subset B$ and read A contained B or $B \supset A$ and read B contains A .
- If $A \subset B$ and $B \subset A$ then $A = B$.

Exp) Let S be the set of all natural numbers; $S = \{1, 2, 3, 4, 5, \ldots \ldots\}$

Define : $A = \{ x : x \quad is\ an\ even\ number, \quad x \in S\}$

And $B = \{ y : y \quad is\ a\ multiple\ of\ 3\ , \quad y \in S\}$

Find : $(A^c\ ,\ B^c\ ,\ A/B\ \ ,\ B/A\ )$

Solution:

$A = \{ 2, 4, 6, 8, 10, 12, 14, 16, \ldots\}$

$B = \{ 3, 6, 9, 12, 15, 18, 21, \ldots\}$

$A^c = \{ 1, 3, 5, 7, 9, \ldots\}$; $A^c = \{ x : x \in S \quad \mathbf{and} \quad x \notin A\}$

$B^c = \{ 1, 2, 4, 5, 7, 8, 10, 11, \ldots\}$

$A/B = \{ x : x \in A \quad \mathbf{and} \quad x \notin B\}$

$A/B = \{2, 4, 8, 10, 14, 16, 20, \ldots\}$

$B/A = \{ 3, 9, 15, 21, 27, \ldots\}$

11-***Union of set***: Let A and B be any two subset of S , the union of A and B , denoted by $(A \cup B)$; is the set of all elements belonging either to A or to B or both.

$(A \cup B) = \{x : either\ x \in A\ or\ x \in B\ or\ both\}$

$x \in (A \cup B) = x \in A \quad or \quad x \in B$

$x \notin (A \cup B) = x \notin A \quad and \quad x \notin B$

12-***Intersection of set***: Let A and B be any two subset of S , The intersection denoted by $(A \cap B)$ is the set of all elements that are in both A and B .

$(A \cap B) = \{x : x \in A \quad and \quad x \in B\ \}$

$x \in (A \cap B) = x \in A \quad and \quad x \in B$

$x \notin (A \cap B) = x \notin A \quad or \quad x \notin B$

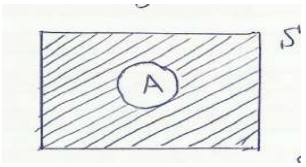
- Two sets A and B are said to be disjoint or mutually exclusive if they have no common elements i.e. :

  $A \cap B = \emptyset$

- **Venn-Diagrams :**

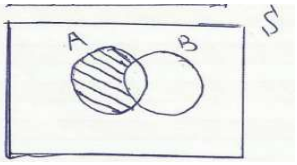
The set operations can be represented by diagrams which are called Venn diagrams .

1- A^c =



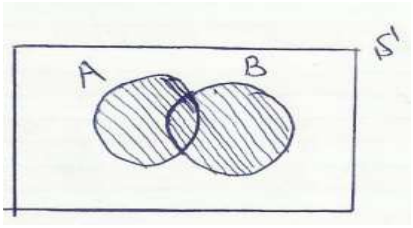
$$A^c = \{ x : x \in S \quad \textbf{and} \quad x \notin A \}$$

2- A/B =



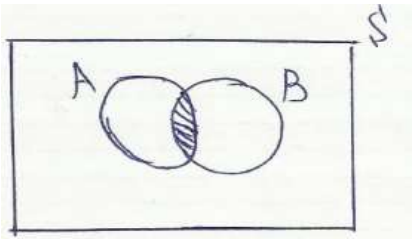
$$A/B = \{ x : x \in A \quad \textbf{and} \quad x \notin B \}$$

3- $A \cup B$ =



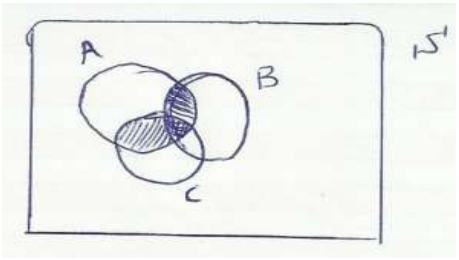
$$(A \cup B) = \{x : x \in A \text{ or } x \in B \text{ or both} \}$$

4- $A \cap B$ =

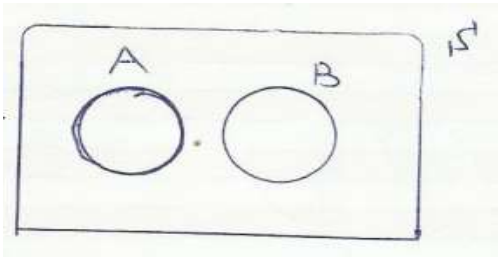


$$(A \cap B) = \{x : x \in A \text{ and } x \in B \}$$

5- $\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C})$ =



6- $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$



1.2- Some Fundamental Theorems بعض النظريات الأساسية

- Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be any subset of $\boldsymbol{S}$

1- Idempotent Laws قانون الثبات

$$\boldsymbol{A} \cup \boldsymbol{A} = \boldsymbol{A} \quad and \quad \boldsymbol{A} \cap \boldsymbol{A} = \boldsymbol{A}$$

2- Associative Laws قانون التبديل

$$\boldsymbol{A} \cup (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cup \boldsymbol{C} = \boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}$$

3- Commutative Laws قانون التساوي

$$\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{B} \cup \boldsymbol{A} \quad and \quad \boldsymbol{A} \cap \boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{A}$$

4- Distributive Laws قانون التوزيع

$$\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cap (\boldsymbol{A} \cup \boldsymbol{C})$$

$$\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{A} \cap \boldsymbol{C})$$

5- Identity Laws قانون الأحادية

$$\boldsymbol{A} \cup \emptyset = \boldsymbol{A} \quad and \quad \boldsymbol{A} \cap \emptyset = \emptyset$$

$$\boldsymbol{A} \cup \boldsymbol{S} = \boldsymbol{S} \quad and \quad \boldsymbol{A} \cap \boldsymbol{S} = \boldsymbol{A}$$

6- Complement Laws قانون المتممة او المكمل

$$\boldsymbol{A} \cup \boldsymbol{A^c} = \boldsymbol{S} \quad and \quad A \cap \boldsymbol{A^c} = \emptyset$$

$$(\boldsymbol{A^c})^{\boldsymbol{c}} = \boldsymbol{A} \quad ; \quad \boldsymbol{S^c} = \emptyset \quad and \quad \emptyset^{\boldsymbol{c}} = \boldsymbol{S}$$

7- De Morgan's Laws قانون دي موركان

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

** To prove the distributive Laws :

1- $\boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

Proof/

Let $x \in A \cup (B \cap C) \rightarrow x \in A \ \ or \ x \in (B \cap C)$

$\rightarrow \ x \in A \ \ or \ (x \in B \ and \ x \in C)$

$\rightarrow \ (x \in A \ \ or \ x \in B) \ and \ (x \in A \ or \ x \in C)$

$\rightarrow \ \big(x \in (A \cup B) \ and \ x \in (A \cup C)\big)$

$\rightarrow \ x \in (\boldsymbol{A \cup B}) \cap (\boldsymbol{A \cup C})$

$\therefore \ \boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$ والعكس صحيح

2- $\boldsymbol{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

Proof/

Let $x \in A \cap (B \cup C) \rightarrow x \in A \ \ and \ x \in (B \cup C)$

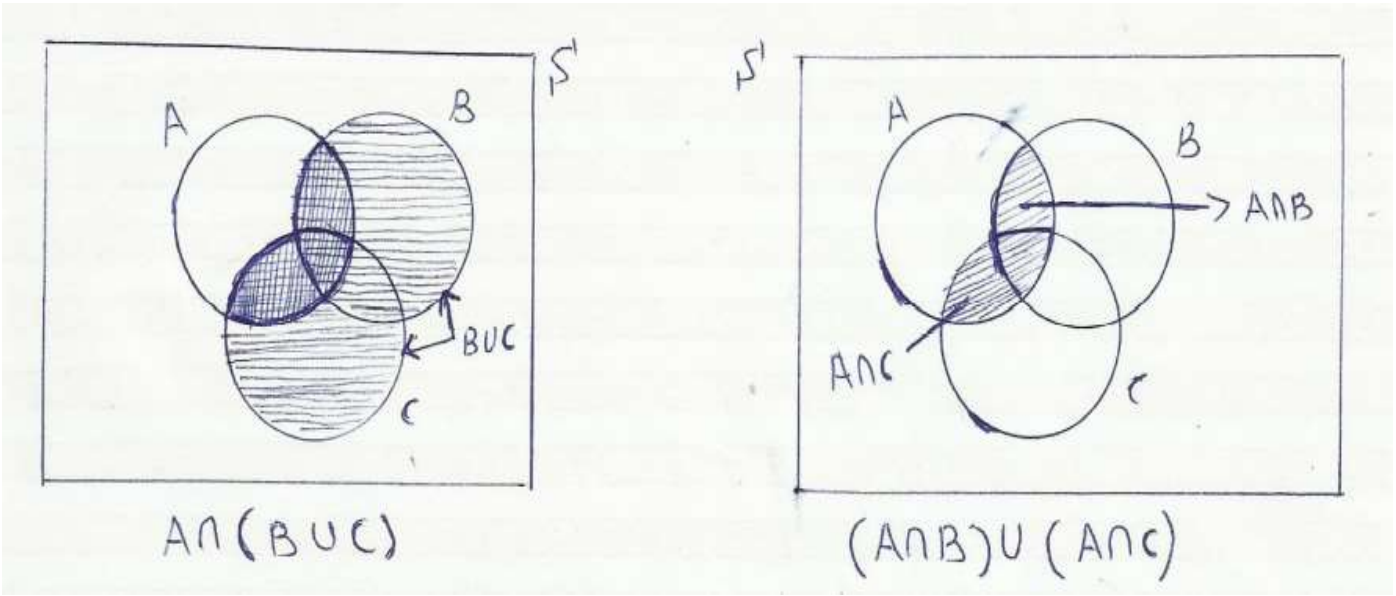
$\rightarrow \ x \in A \ \ and \ (x \in B \ or \ x \in C)$

$\rightarrow \ (x \in A \ \ and \ x \in B) \ or \ (x \in A \ and \ x \in C)$

$\rightarrow \ \big(x \in (A \cap B) \ or \ x \in (A \cap C)\big)$

$\rightarrow \ x \in (\boldsymbol{A \cap B}) \cup (\boldsymbol{A \cap C})$

$\therefore \ \boldsymbol{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$



** To prove the De Morgan's Laws :

1- $(A \cup B)^c = A^c \cap B^c$

Proof/

Let $x \in (A \cup B)^c \rightarrow x \in S \text{ and } x \notin (A \cup B)$

$\rightarrow x \in S \text{ and } (x \notin A \text{ and } x \notin B)$

$\rightarrow (x \in S \text{ and } x \notin A) \text{ and } (x \in S \text{ and } x \notin B)$

$\rightarrow x \in A^c \text{ and } x \in B^c$

$\rightarrow x \in (A^c \cap B^c)$

$\therefore (A \cup B)^c \subset A^c \cap B^c$

برهان الطرف الثاني

$A^c \cap B^c = (A \cup B)^c$

Proof/

Let $x \in A^c \cap B^c \rightarrow x \in A^c \text{ and } x \in B^c$

$\rightarrow (x \in S \text{ and } x \notin A) \text{ and } (x \in S \text{ and } x \notin B)$

$\rightarrow x \in S \text{ and } (x \notin A \text{ and } x \notin B)$

$\rightarrow x \in S \text{ and } x \notin (A \cup B)$

$\rightarrow x \in (A \cup B)^c$

$\therefore A^c \cap B^c \subset (A \cup B)^c$

2- $(A \cap B)^c = A^c \cup B^c$

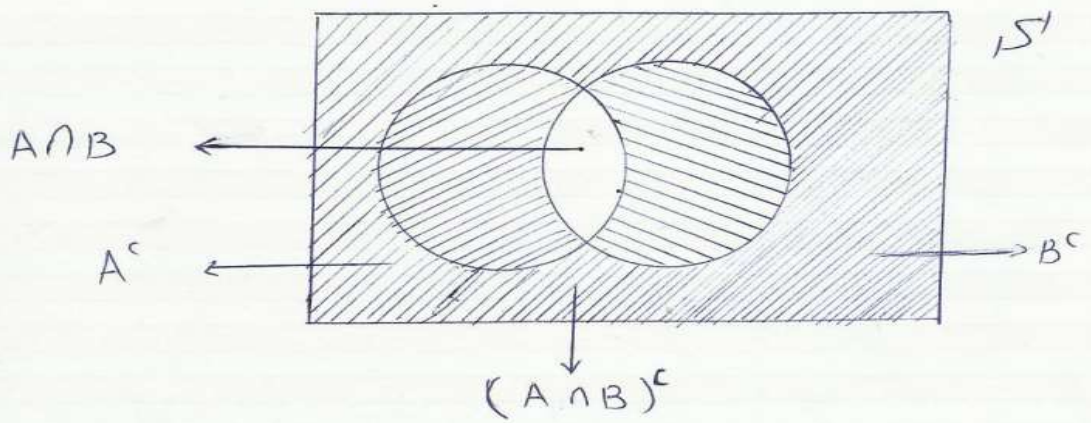
Proof/ Let $x \in (A \cap B)^c \rightarrow x \in S'$ and $x \notin (A \cap B)$

$\rightarrow x \in S'$ and $(x \notin A$ or $x \notin B)$

$\rightarrow (x \in S'$ and $x \notin A)$ or $(x \in S'$ and $x \notin B)$

$\rightarrow x \in A^c$ or $x \in B^c$

$\rightarrow x \in (A^c \cup B^c)$



* ملاحظة مهمة عن اتحاد

1- $x \in A \cup B \rightarrow x \in A$ or $x \in B$

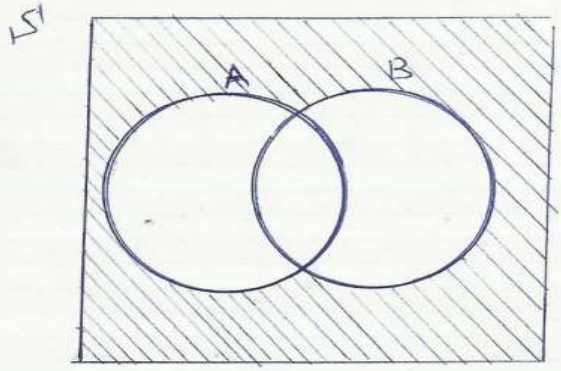
$x \notin A \cup B \rightarrow x \notin A$ and $x \notin B$

تقاطع

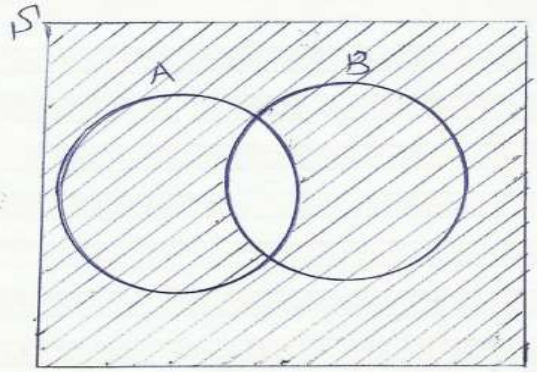
2- $x \in A \cap B \rightarrow x \in A$ and $x \in B$

$x \notin A \cap B \rightarrow x \notin A$ or $x \notin B$

ويمكن اثبات $(A \cup B^c) = A^c \cap B^c$ بأحد



$(A \cup B)$



$(A \cap B)$

3 - $x \in (A/B) \rightarrow x \in A$ and $x \notin B$

4- $x \in A^c \rightarrow x \in S$ and $x \notin A$

**** Prove that :**

a) $A/B = A \cap B^c$

b) $B/A = B \cap A^c$

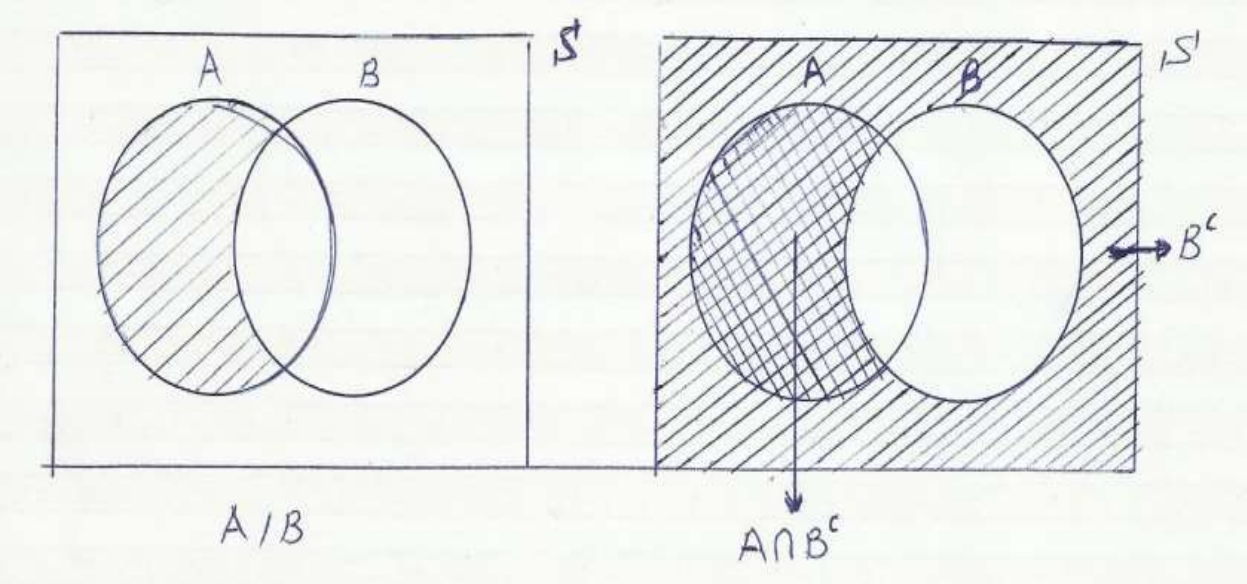
سيتم الاثبات بطريقتين

1- باستخدام الرسم .

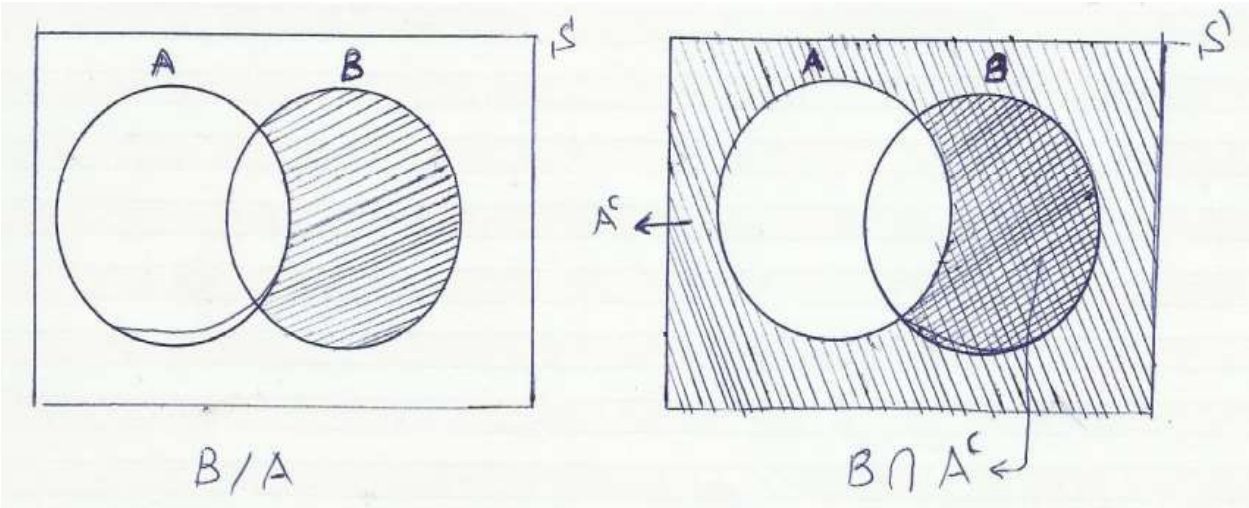
2- باستخدام البرهان .

.....الاثبات بالرسم :

a) $A/B = A \cap B^c$



b) $\boldsymbol{B/A} = \boldsymbol{B} \cap \boldsymbol{A^c}$



الاثبات باستخدام البرهان

a) $\boldsymbol{A/B} = \boldsymbol{A} \cap \boldsymbol{B^c}$

- Let $x \in (A/B) \rightarrow x \in A$ and $x \notin B$
$\rightarrow x \in A$ and $(x \in S$ and $x \notin B)$
$\rightarrow x \in A$ and $x \in B^c$
$\rightarrow x \in (A \cap B^c)$
$\rightarrow A/B \subset (A \cap B^c)$ ..... (1)
- Let $x \in (A \cap B^c) \rightarrow x \in A$ and $x \in B^c$
$\rightarrow x \in A$ and $(x \in S$ and $x \notin B)$
$\rightarrow x \in A$ and $x \notin B$
$\rightarrow x \in (A/B)$
$\rightarrow (A \cap B^c) \subset A/B$ ..... (2)

b) $\boldsymbol{B/A = B \cap A^c}$

let $x \in (B \ / A) \quad \rightarrow \quad x \in B \ and \ x \notin A$.

$\rightarrow \quad x \in B \ and \ (x \in S \ and \ x \notin A)$

$\rightarrow \quad x \in B \ and \ x \in A^c$

$\rightarrow \quad x \in (B \cap A^c)$

$\rightarrow \quad (B \ / A) \subset (B \cap A^c)$

برهان الطرف الاخر

let $x \in (B \cap A^c) \quad \rightarrow \quad x \in B \ and \ x \in A^c$.

$\rightarrow \quad x \in B \ and \ (x \in S \ and \ x \notin A)$

$\rightarrow \quad x \in B \ and \ x \notin A$

$\rightarrow \quad x \in (B \ / A)$

$\rightarrow \quad (B \cap A^c) \subset (B \ / A)$

$\therefore \quad (B \ / A) = (B \cap A^c)$

** Homework :-

1- $\boldsymbol{A \cap (B \ / \ C) = (A \cap B) \ / \ (A \cap C)}$

2- $\boldsymbol{A \cup B = A \cup (B \cap A^c)}$

1.3- Sequences and limits المتتابعات والغايات

We have already defined union and intersection of two sets ; these definitions extend to more than two sets .

To distinguish (تميز) between the sets in a collection of subsets of S , we assign names to the form of subscripts

لحد الان تم تعريف الاتحاد والتقاطع لمجموعتان فإذا كان لدينا اكثر من مجموعتان ، يجب ان نتعرف على المتتابعات التي يمكن تعريفها بانها عبارة عن مجموعة من المجاميع الجزئية معرفة من مجموعة شاملة .

Let I be an index set and $\{A_r : r \in I\}$ be a collection of subset of S indexed by I , which is called a Sequence of sets .

1- The Union of Sequences of sets :

$$\bigcup_{r\in I} A_r = \{ x : x \in S \, ; x \in A_r \;\; for \; some \;\; r \in I\}$$

$$\bigcup_{r\in I} A_r = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \dots \cup A_n$$

$$\rightarrow x \in \bigcup_{r\in I} A_r \;\; if \; and \; only \; if \; x \in A_r \;\; for \; some \;\; r \in I$$

$$\rightarrow x \notin \bigcup_{r\in I} A_r \;\; if \; and \; only \; if \; x \notin A_r \;\; for \; all \;\; r \in I$$

2- The Intersection of Sequences of sets :

$$\bigcap_{r\in I} A_r = \{x : x \in S; \; x \in A_r \;\; for \; all \;\; r \in I\}$$

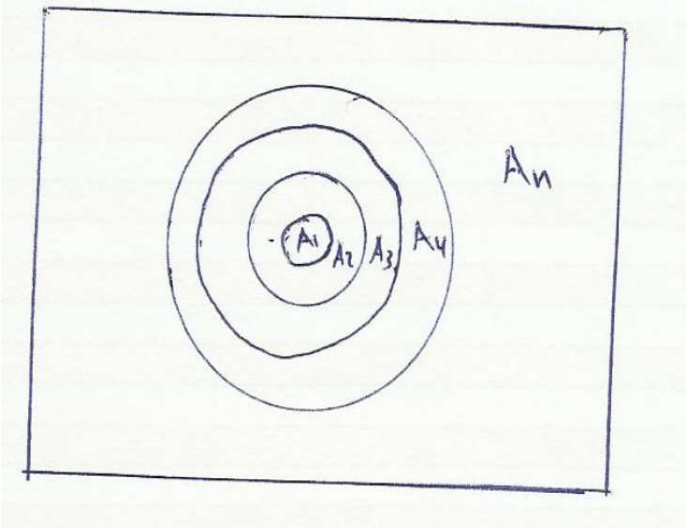
$$\bigcap_{r\in I} A_r = A_1 \cap A_2 \cap A_3 \cap A_4 \cap \dots \cap A_r$$

$$\rightarrow x \in \bigcap_{r\in I} A_r \; if \; and \; only \; if \; x \in A_r \;\; for \; all \;\; r \in I$$

$$\rightarrow x \notin \bigcap_{r\in I} A_r \; if \; and \; only \; if \; x \notin A_r \;\; for \; some \;\; r \in I$$

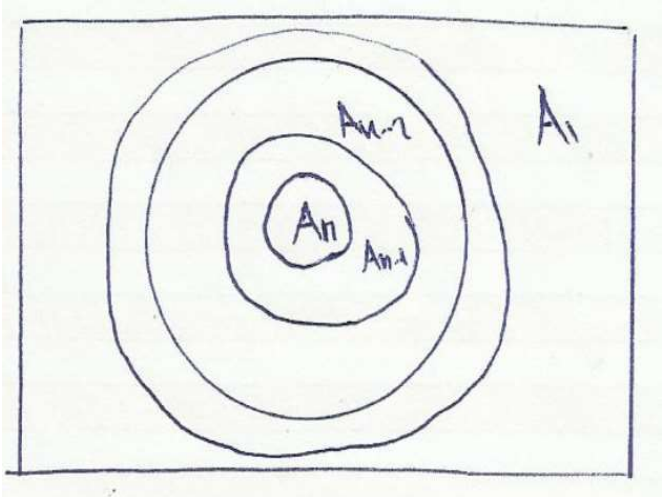
3- Increasing sequences المتتابعات المتزايدة

$A_1 < A_2 < A_3 < A_4 < \cdots ;\ \bigcup_{r\in I} A_r = A_n\ ;\ \bigcap_{r\in I} A_r = A_1$



4- Decreasing sequences المتتابعات المتناقصة

$A_n < A_{n-1} < A_{n-2} < A_{n-4} < \cdots A_i\ ;\ \bigcup_{r\in I} A_r = A_1\ ;\ \bigcap_{r\in I} A_r = A_n$



** De Morgan's Theorem :-

Let I be an index set and $\{A_r : r \in I\}$ a sequence of subset of S indexed by I. Then :-

1- $(\cup_i A_i)^c = \cap_i A_i^c$

2- $(\cap_i A_i)^c = \cup_i A_i^c$

Proof (1)/

Let $x \in (\cup_i A_i)^c \rightarrow x \in S$ and $x \notin \cup_i A_i$

$\rightarrow (x \in S$ and $x \notin A_i)$ for all $i \in I$

$\rightarrow x \in A_i^c$ for all $i \in I$

$\rightarrow x \in \cap_i A_i^c$

$\rightarrow (\cup_i A_i)^c \subset \cap_i A_i^c$

برهان الطرف الاخر

Let $x \in \cap_i A_i^c \rightarrow x \in A_i^c$ for all $i \in I$

$\rightarrow x \in S$ and $x \notin A_i$ for all $i \in I$

$\rightarrow x \in S$ and $x \notin \cup_i A_i$

$\rightarrow x \in (\cup_i A_i)^c$

$\rightarrow \cap_i A_i^c \subset (\cup_i A_i)^c$

$\therefore (\cup_i A_i)^c = \cap_i A_i^c$

Proof (2)/

Let $x \in (\cap_i A_i)^c \rightarrow \boldsymbol{x \in S}$ *and* $x \notin \cap_i A_i$

$\rightarrow (\boldsymbol{x \in S}$ *and* $x \notin A_i)$ $for\ some\ \ i \in I$

$\rightarrow \boldsymbol{x} \in {A_i}^c$ $for\ some\ \ i \in I$

$\rightarrow x \in \bigcup_i {A_i}^c$

$\rightarrow (\bigcap_i A_i)^{\boldsymbol{c}} \subset \bigcup_i {A_i}^c$

برهان الطرف الاخر

Let $x \in \bigcup_i {A_i}^c \rightarrow \boldsymbol{x} \in {A_i}^c$ $for\ some\ \ i \in I$

$\rightarrow (\boldsymbol{x \in S}$ *and* $x \notin A_i)$ $for\ some\ \ i \in I$

$\rightarrow \boldsymbol{x \in S}$ *and* $x \notin \bigcap_i A_i$

$\rightarrow x \in (\bigcap_i A_i)^{\boldsymbol{c}}$

$\rightarrow \bigcap_i {A_i}^c \subset (\bigcup_i A_i)^{\boldsymbol{c}}$

$\therefore (\cap_i A_i)^{\boldsymbol{c}} = \cup_i {A_i}^c$

**** Homework :-**

1- $\boldsymbol{B} \cup (\cap_i A_i) = \cap_i (\boldsymbol{B} \cup A_i)$

2- $\boldsymbol{B} \cap (\cup_i A_i) = \cup_i (\boldsymbol{B} \cap A_i)$

** Power of The Set **قوة المجموعة او صلاحية المجموعة**

Power of the set A : $P^N(A)$ is the set of all subsets of the set A, it contains 2^n subset, where n is number of elements in A.

هي مجموعة كل المجاميع الجزئية المتكونة من المجموعة A وتحتوي على 2^n من المجاميع الجزئية وحيث ان **n** هو عدد العناصر الموجودة في A .

Exp) If $A = \{1,2,3\}$; Find $P^N(A)$

Sol/

$$n = 3 \quad ; \quad 2^n = 2^3 = 8$$

$$\therefore P^N(A) = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},A,\emptyset\}$$

** Partition of a set A **تجزئة او تقسيم المجموعة** A

Partition of a set in to a non-empty subset which are disjoint and whose union is A.

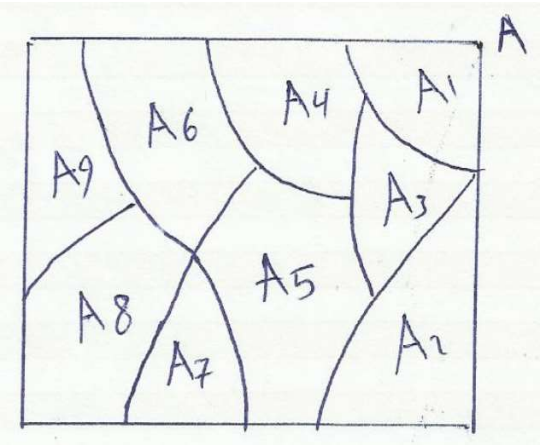
تجزئة المجموعة A الى مجاميع جزئية غير خالية من العناصر بحيث تكون غير مرتبطة وناتج اتحادها المجموعة A

1- $A_i \cap A_j = \emptyset$; $A_1 \cap A_2 = \emptyset$

$i \neq j$; $A_2 \cap A_8 = \emptyset$

2- $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_9 = A$

$$\therefore \bigcup_{i=1}^{9} A_i = A$$



1.5- Problems بعض مسائل الفصل

7 - Let A, B and C be any three subsets of S, Prove that

a-/ $A \cap (B/C) = (A \cap B) \cap (A/C)$

Proof/: let $x \in A \cap (B/C) \rightarrow x \in A$ and $x \in (B/C)$

$\rightarrow x \in A$ and $(x \in B$ and $x \notin C) \rightarrow$

$\rightarrow (x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C)$

$\rightarrow x \in (A \cap B)$ and $x \in (A/C) \rightarrow x \in (A \cap B) \cap (A/C)$

8- let $A = \{0, 1, 2\}$, state whether or not each statement is correct.

i- $\{0\} \notin A$ F. ; iii- $2 \in A$, $2 \subset A$ T.

ii- $\{0, 2\} \subset A$ T. ; iv- $\{2, 1, 0\} \subset A$ T.

9- let $A = \{w : 2w = 6\}$, $B = \{5\}$ and $C = \{x : 3x = 9\}$

which of these sets are equal ?

$A = \{2w = 6\} \Rightarrow (w = 3)$, $B = \{5\}$, $C = \{3x = 9\} \Rightarrow (x = 3)$

$\therefore A = C$ where $w = x = 3$.

10- let $A = \{1, 3, 4, 5\}$, $B = \{3, 5\}$, $C = \{a, b, 3, 5, 6\}$ and $D = \{\emptyset\}$.

State whether each statement is true or false.

i- $B \subset C$, T

ii- $A \subset C$, F

iii- $D \subset A$, T

iv- $A \cup B = A$, T

v- $A \cap B = \emptyset$, T

vi- $B/C = \emptyset$, T

vii- $A/B = \{1, 5\}$, F

11- Let $S=\{1,2,3,4,5,6,7\}$; $A=\{1,2,3,4,5\}$; $B=\{1,3,5,7\}$ and $C=\{2,5,6,7\}$; Find:-

i- $A \cup C = \{1,2,3,4,5,6,7\} = S$

ii- $A \cap B = \{1,3,5\}$

iii- $C/B = \{2,6\}$

iv- $A \cap C^c = \{1,3,4\}$

v- $(A/C)^c = \{2,5,6,7\}$; $A/C = \{1,3,4\}$

vi- $(A/B^c)^c = \{2,4,6,7\}$; $B^c = \{2,4,6\}$, $A/B^c = \{1,3,5\}$

vii $= (C \cap C^c)^c = (S) = \{1,2,3,4,5,6,7\}$; $(C \cap C^c) = \emptyset$ and $\emptyset^c = S$

ملاحظة : تعريف

$A * B = \{x, y;\ x \in A \text{ and } y \in B\}$

$\therefore A * (B \cap C) = (A * B) \cap (A * C)$

Ex/ $A = \{1,2,3\}$; $B = \{a,b\}$

$A * B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$B * A = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

16- Prove that for any indexed class $\{A_i : i \in I\}$ and any Subset B;

a - $B \cup (i \cap A_i) = i \cap (B \cup A_i)$; b - $B \cap (i \cup A_i) = i \cup (B \cap A_i)$.

Proof/

a - let $x \in B \cup (i \cap A_i) \rightarrow x \in B$ or $x \in (i \cap A_i)$

$\rightarrow x \in B$ or $(x \in A_i$ for all $i \in I)$

$\rightarrow ((x \in B$ or $x \in A_i)$ for all $i \in I)$

$\rightarrow x \in (B \cup A_i)$ for all $i \in I$

$\rightarrow i \cap (B \cup A_i)$

وبنفس الاسلوب بالامكان برهنة الطرف الثاني ؟

B - H.W.

Chapter two

- Techniques of Counting –

تقنية العدّ :

من اجل اجراء عملية الترتيب واجراء عملية العد والحساب لابد من التعرف على كيفية تكوين مخطط الشجرة لكي يساعدنا في عملية الاحتساب والعد .

Tree Diagrams مخطط الشجرة

A tree diagram provides an organized way of listing the arrangements so that none is missed . Note that the tree diagram takes order into account .

1- مخطط الشجرة يوفر طريق لترتيب المجاميع الى أي شيء مطلوب ترتيبه .

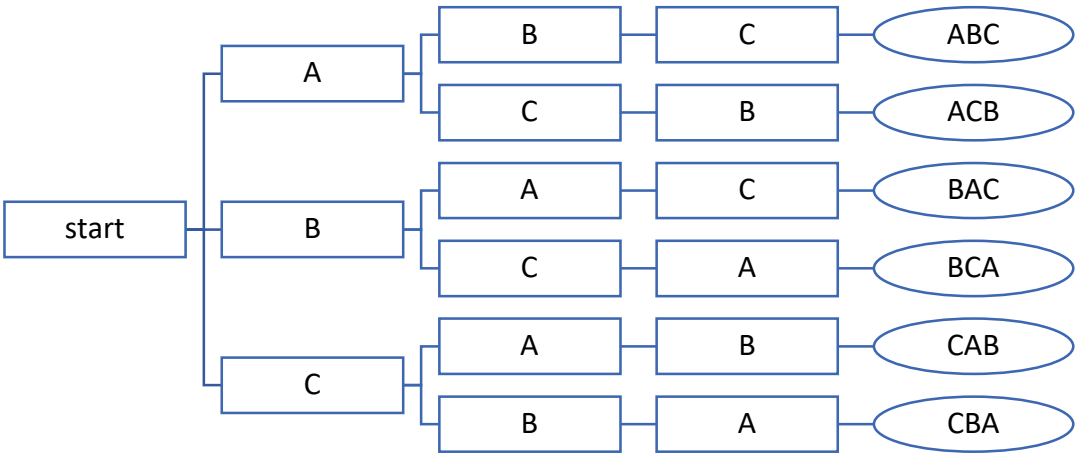
2- مخطط الشجرة يساعدنا في ترتيب قائمة من الأشياء بدون ان نختار أي ترتيب حيث يأخذ الترتيب بنظر الاعتبار .

In another words : a tree diagram is a device used to enumerate all the possible outcomes of a sequence of experiments , where each experiment can occur in a finite number of ways.

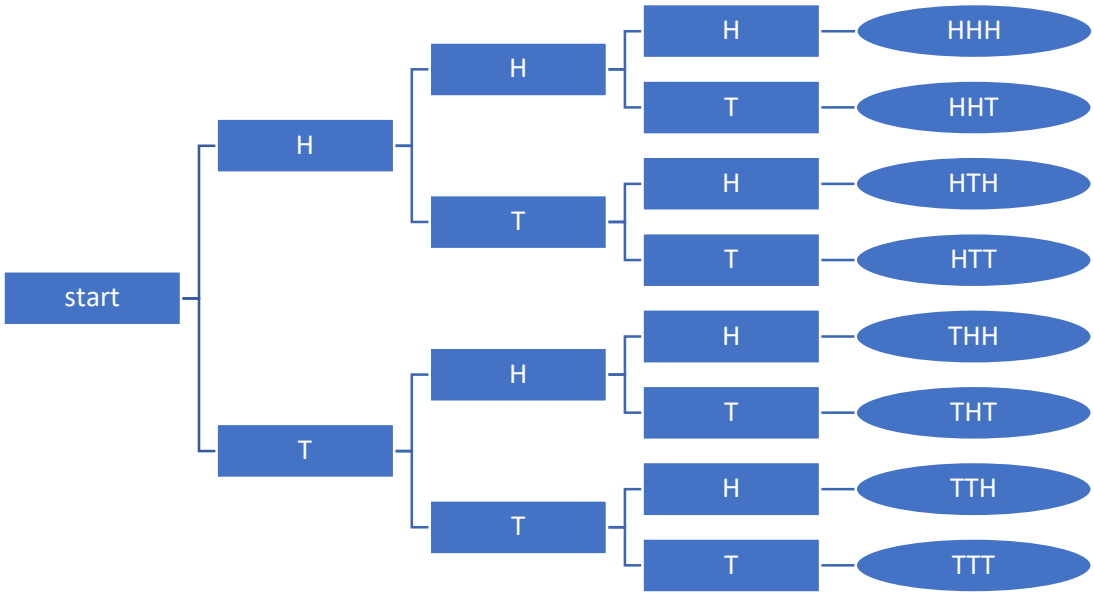
بعبارة أخرى ، مخطط الشجرة عبارة عن مخطط يستخدم لوضع جميع النتائج الممكنة لاي تجربة ويمكن ظهور هذه التجربة بعدد محدد من الطرق .

Exp)

In how many ways can 3 books denoted by A , B and C be arrange in order on a shelf ?



Exp) Tossing 3 coins (H – Head-*الصورة*) (T – Tail-*الكتابة*)



There are methods for the counting process represented by:

2-1) Arrangements

If n objects are taken out of the total number of objects to form groups with attention being paid to the order of objects in each group , then these groups are called "arrangements" .

اذا اخذنا **n** من الأشياء والمخرجة من مجموعة من الأشياء ونريد ان نرتبها الى مجاميع وهذه المجاميع تسمى "تراتيب"

وتأخذ التراتيب حالتان :

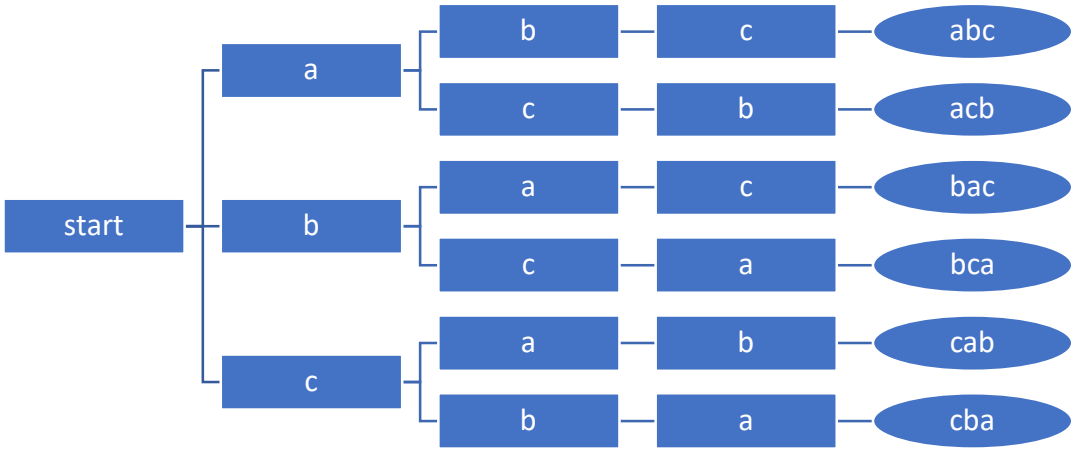
1_ Repetition is allowed : تكرار مسموح

2_ Repetition is not allowed : تكرار غير مسموح

Exp)

In how many ways can the letters a , b and c be arranged in a row , with 3_digit ; If repetition is not allowed :

3 2 1 $= 3 \times 2 \times 1 = 6$ عدد الطرق الكلية

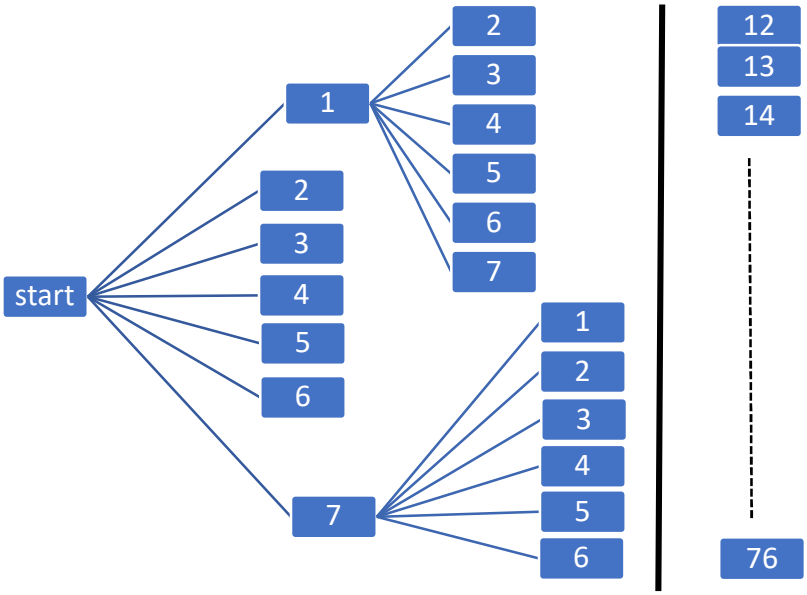


Exp)

How many 2_digit numbers can be formed from the seven digit numbers {1,2,3,4,5,6,7} if repetition is not allowed :

Sol/

7 6 $= 7 \times 6 = 42$ ways



كم عدد فردي (odd) يمكن تكوينه من هذه الاعداد في 2_digit

يتم احتساب الاعداد الفردية أولا ووضعها في موقها فيمثل عدد الأرقام باستبعاد الاعداد الزوجية .

الأرقام من المثال السابق .....

_ $\underline{6}\ \underline{4} = 6 \times 4 = 24$ ways

كم عدد زوجي (even) يمكن تكوينه من هذه الاعداد في 2_digit

_ $\underline{6}\ \underline{3} = 6 \times 3 = 18$ ways

* if repetition is allowed :

Exp)

How many 2_digit numbers can be formed from {1,2,3,4,5,6,7} ; if repetition is allowed :

Sol/

_ $\underline{7}\ \underline{7} = 7 \times 7 = 49$ ways

_ $\underline{7}\ \underline{4} = 7 \times 4 = 28$ ways اعداد فردية وتكرار مسموح

_ $\underline{7}\ \underline{3} = 7 \times 3 = 21$ ways اعداد زوجية وتكرار مسموح

Exp)

How many 4_digit numbers can be formed from {1,2,4,5,6,7,9} ; find if ?

1-repetition is not allowed

2- repetition is allowed

Sol/

_ $\underline{7}\ \underline{6}\ \underline{5}\ \underline{4} = 7 \times 6 \times 5 \times 4 = 840$ ways تكرار غير مسموح

_ $\underline{7}\ \underline{7}\ \underline{7}\ \underline{7} = 7 \times 7 \times 7 \times 7 = 2401$ ways تكرار مسموح

____ عدد زوجي (even)

_ 6 5 4 3 $= 6 \times 5 \times 4 \times 3 = 360$ ways تكرار غير مسموح

_ 7 7 7 3 $= 7 \times 7 \times 7 \times 3 = 1029$ ways تكرار مسموح

____ عدد فردي (odd)

_ 6 5 4 4 $= 6 \times 5 \times 4 \times 4 = 480$ ways تكرار غير مسموح

_ 7 7 7 4 $= 7 \times 7 \times 7 \times 4 = 1372$ ways تكرار مسموح

From the previous example: من المثال السابق

__ how many of them are start with 5 and end with 9 .

_ 1 5 4 1 $= 1 \times 5 \times 4 \times 1 = 20$ ways تكرار غير مسموح

_ 1 7 7 1 $= 1 \times 7 \times 7 \times 1 = 49$ ways تكرار مسموح

Exp)

How many 3_digit numbers can be formed from {0,1,2,3,4,5,6,7,8,9} ; find if ?

1-repetition is not allowed

2- repetition is allowed

Sol/

_ 9 9 8 $= 9 \times 9 \times 8 = 648$ ways تكرار غير مسموح

_ 9 10 10 $= 9 \times 10 \times 10 = 900$ ways تكرار مسموح

__ How many of them are odd number :

_ 8 8 5 $= 8 \times 8 \times 9 = 320$ ways تكرار غير مسموح

_ 9 10 5 $= 9 \times 10 \times 5 = 450$ ways تكرار مسموح

__ How many of them are even number :

هناك حالتان للحل في حالة الرقم زوجي عند وجود الصفر في مجموعة الاعداد :

When repetition is not allowed:

a_ $\underline{8}\ \underline{8}\ \underline{4} = 8 \times 8 \times 4 = 256$ ways

b_ $\underline{9}\ \underline{8}\ \underline{1} = 9 \times 8 \times 1 = 72$ ways

$= (256 + 72) = 328$ ways

When repetition is allowed:

a_ $\underline{9}\ \underline{10}\ \underline{4} = 9 \times 10 \times 4 = 360$ ways

b_ $\underline{9}\ \underline{10}\ \underline{1} = 9 \times 10 \times 1 = 90$ ways

$= (360 + 90) = 450$ ways

Exp)

How many 3_digit numbers can be formed from {0,2,4,5,7} ; find if ?

1-repetition is not allowed

2- repetition is allowed

Sol/

_ $\underline{4}\ \underline{4}\ \underline{3} = 4 \times 4 \times 3 = 48$ ways تكرار غير مسموح

_ $\underline{4}\ \underline{5}\ \underline{5} = 4 \times 5 \times 5 = 100$ ways تكرار مسموح

__ How many of them odd number :

_ $\underline{3}\ \underline{3}\ \underline{2} = 3 \times 3 \times 2 = 18$ ways تكرار غير مسموح

_ $\underline{4}\ \underline{5}\ \underline{2} = 4 \times 5 \times 2 = 40$ ways تكرار مسموح

__ How many of them are even number :

هناك حالتان للحل في حالة الرقم زوجي عند وجود الصفر في مجموعة الاعداد :

When repetition is not allowed:

a_ $\underline{3}\ \underline{3}\ \underline{2} = 3 \times 3 \times 2 = 18$ ways

b_ $\underline{4}\ \underline{3}\ \underline{1} = 4 \times 3 \times 1 = 12$ ways

$= (18 + 12) = 30$ ways

When repetition is allowed:

a_ $\underline{4}\ \underline{5}\ \underline{2} = 4 \times 5 \times 2 = 40$ ways

b_ $\underline{4}\ \underline{5}\ \underline{1} = 4 \times 5 \times 1 = 20$ ways

$= (40 + 20) = 60$ ways

مثال)

كم لوحة سيارة يمكن تكوينها اذا كانت تتكون من خمسة مواقع ، اول موقعين من الحروف والثلاثة من الأرقام ؟

الحل / عدد الحروف 26 ، وعدد الأرقام 0,1,2,...,9

When repetition is not allowed:

_ $\underline{26}\ \underline{25}\ \underline{9}\ \underline{9}\ \underline{8} = 26 \times 25 \times 9 \times 9 \times 8 = 421200$ ways

When repetition is allowed:

_ $\underline{26}\ \underline{26}\ \underline{9}\ \underline{10}\ \underline{10} = 26 \times 26 \times 9 \times 10 \times 10 = 604400$ ways

ملاحظة :

يتم استخدام طريقة المواقع في الحلات الاتية :

1- إيجاد الاعداد الزوجية او الفردية .

2- اذا طلب ايجاد اقل رقم او اكبر رقم من بين مجموعة من الأرقام .

3- اذا ابتدأ برقم معين او انتهى برقم معين .

2_2) Permutations طريقة التباديل

_ Definition :-

An arrangement of a set of n different objects in a given order is called a permutation of the objects (taken all at a time) .

The arrangement of any $r \leq n$ of these objects in a given order is called a permutation of n objects taken r at a time and is denoted by P_r^n or $P(n,r)$.

ملاحظة مهمة : عند استخدام طريقة التباديل يكون الترتيب مهم بالحساب .

$P_r^n = \frac{n!}{(n-r)!}$; $r \leq n$; $0 \leq r \leq n$

1- If $r = n \rightarrow P_r^n = n!$

Where $P_n^n = \frac{n!}{(n-n)!} = \frac{n!}{(0)!} = \frac{n!}{1} = n!$

2- If $r > n \rightarrow P_r^n = 0$

3- If $r = 0 \rightarrow P_r^n = 1$

Where $P_0^n = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

4- If repetition is allowed $\rightarrow P_r^n = n^r$

5- $n = n_1 + n_2 + n_3 + \cdots + n_k \rightarrow P_{n_1,n_2,n_3,\ldots,n_k}^n = \frac{n!}{n_1! * n_2! * n_3! * \ldots * n_k!}$

6- The total number of arrangement of n different objects around a circle is $(n-1)!$

7- The number of permutation of n different objects taken all at a time is $n!$.

(in a row على صف واحد)

Exp) If repetition are not allowed :

1_ How many 3_digit numbers can be formed from the six digits {2,3,4,5,6,7}.

2_ How many of these are less than 400 .

3_ how many are even .

4_ how many are odd .

5_ how many are multiple of 5 .

Sol/

1_ $\underline{6}\ \underline{5}\ \underline{4} = 6 \times 5 \times 4 = 120$ ways

0r

$$P_r^n = \frac{n!}{(n-r)!} \rightarrow P_3^6 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\times5\times4\times3!}{3!} = 120$$

ملاحظة : اذا ذكر بالسؤال شرط (إيجاد عدد فردي او زوجي) او عبارة (اقل من او اكبر من عدد معين) لا يمكن حل السؤال إلا بطريقة المواقع .

2_ $\underline{2}\ \underline{5}\ \underline{4} = 2 \times 5 \times 4 = 40$ ways

3_ $\underline{5}\ \underline{4}\ \underline{3} = 5 \times 4 \times 3 = 60$ ways

4_ $\underline{5}\ \underline{4}\ \underline{3} = 5 \times 4 \times 3 = 60$ ways

5_ $\underline{5}\ \underline{4}\ \underline{1} = 5 \times 4 \times 1 = 20$ ways

Exp) In how many ways can a party of 7 persons arrange them selves .

1_ In a row of chairs .

2_ A round a circular table .

Sol/

1_ $n! = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

2_ $(n-1)! = (7-1)! = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Exp)

1_ In how many ways can 3 boys and 2 girls sit in a row .

2_ In how many ways can they sit in a row if just the boys are sit together .

3_ In how many ways can they sit in a row if just the girls are sit together .

Sol/

1_ $n! = (3+2)! = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \ \ ways$

2_ $3! \times 3! = 6 \times 6 = 36 \ \ ways$.

3_ $2! \times 4! = 2 \times 1 \times 4 \times 3 \times 2 \times 1 = 48 \ ways$.

Exp) Find the number of ways in which three of ten new movies are be ranked first , second and third by movies critics (لجنة تحكيمية).

Sol/ Let $n = 10$; $r = 3$

$$P_r^n = \frac{n!}{(n-r)!} \rightarrow P_3^{10} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = 720 \ ways$$

Exp) what is the number of ways in which 6 persons can seated in a row if a certain of them must sit side by side .

Sol/

1_ $n! = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \ ways$

2_ 2 sit side by side $\rightarrow \ 2! \times 5! = \ 240 \ ways$

3_ 3 sit side by side $\rightarrow \ 3! \times 4! = \ 144 \ ways$

Exp) In how many ways can eight teaching assistant be assigned to eight classes of a course in probability ?

Sol/ Let $n = 8$; $r = n = 8$

$$P_r^n = n! = P_8^8 = 8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320 \ \ ways$$

Exp) How many different words can be formed from the 8 letter is (REMEMBER) ?

Sol/

$$P^{n}_{n_1,n_2,n_3,\dots,n_k} \quad ; \quad R = 2 \; ; \; E = 3 \; ; \; M = 2 \; ; \; B = 1$$

$$P^{8}_{2,3,2,1} = \frac{n!}{n_1!*n_2!*n_3!*n_4!} = \frac{8!}{2!*3!*2!*1!} = 1680 \; ways$$

Exp) How many different words can be formed from (MISSISSIPPI) ?

Sol/

$$P^{n}_{n_1,n_2,n_3,\dots,n_k} \quad ; \quad M = 1 \; ; \; S = 4 \; ; \; P = 2 \; ; \; I = 4$$

$$P^{11}_{1,4,2,4} = \frac{n!}{n_1!*n_2!*n_3!*n_4!} = \frac{11!}{1!*4!*2!*4!} = 34650 \; ways$$

2_3) Combinations طريقة التوافيق

Definition :-

A combination is a selection of objects considered without regard to there order . الترتيب غير مهم

The number of combinations of a set of n different objects , taken r at a time is :

$$C_r^n = \frac{n!}{r!(n-r)!} \; ; \quad 0 \le r \le n$$

1_ $If \; r > n \; ; \rightarrow C_r^n = 0$

2_ $If \; r = n \; and \; if \; r = 0 \; ; \rightarrow C_r^n = \frac{n!}{n! \times 0!} = 1$

3_ $C_n^n = C_{n-r}^n$

4_ $If \; n = n_1 + n_2 + n_3 + \cdots + n_k \rightarrow C_{n_1,n_2,n_3,\ldots,n_k}^n = \frac{n!}{n_1! * n_2! * n_3! * \ldots * n_k!}$

Exp) In how many ways can a reader select 3 books without regard to their order from a set of 4 different books denoted by A , B , C , and D ?

Sol/ ABC , ABD , ACD , BCD

So that $C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_3^4 = \frac{4!}{3!(4-3)!} = \frac{4!}{3! \times 1!} = \frac{4 \times 3!}{3!} = 4 \; ways$

Exp) The 3 men can be chosen from the 7 men and the 2 women can be chosen from the 5 women, Hence the committee can be chosen in ?

Sol/

$$C_3^7 \times C_2^5 = \frac{7!}{3!(7-3)!} \times \frac{5!}{2!(5-2)!} = \frac{7!}{3! \times 4!} \times \frac{5!}{2! \times 3!} = 350 \; ways$$

Exp) A student is to answer 8 out of 10 questions on an example .

1_ How many choice has he ?

2_ How many choice has if he must answer the first 3 questions ?

3_ How many choice has if he must answer at least 4 of the first 5 questions ?

4_ How many choice has if he must answer all the first 5 questions ?

Sol/

1_ The 8 questions can be selected in :

$$C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_8^{10} = \frac{10!}{8!(10-\ )!} = \frac{10!}{8! \times 2!} = 45 \ ways$$

2_ if he must answer the first 3 questions ?

$$C_r^n = \frac{n!}{r!(n-r)!} \rightarrow C_5^7 = \frac{7!}{5!(7-5)!} = \frac{7!}{5! \times 2!} = 21 \ ways$$

3_ if he must answer at least 4 of the first 5 questions ?

$$C_4^5 \times C_4^5 + C_5^5 \times C_3^5$$

4_ if he must answer all the first 5 questions ?

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \times 2!} = 10 \ ways$$

Exp) In how many ways can a set of balls be selected from 8 white and 6 red balls such that there will be 3 white and 2 red balls ?

Sol/

8W balls , 6R balls

$$C_3^8 \times C_2^6 = \frac{8!}{3!(8-3)!} \times \frac{6!}{2!(6-2)!} = 56 \times 15 = 840 \ ways$$

Exp)A class contain 9 boys and 3 girls :

1_ In how many ways can the teacher choose a committee of 4 .

2_ How many of them will contain at least one girl .

3_ How many of them will contain exactly one girl .

Sol/

1_ $C_4^{12} = \frac{12!}{4!(12-4)!} = \frac{12!}{4! \times 8!} = 495 \ ways$

2_ $C_1^3 \times C_3^9 + C_2^3 \times C_2^9 + C_3^3 \times C_1^9 =$

$\frac{3!}{1!\times 2!} \times \frac{9!}{3!\times 6!} + \frac{3!}{2!\times 1!} \times \frac{9!}{2!\times 7!} + \frac{3!}{3!\times 0!} \times \frac{9!}{1!\times 8!} = 369 \ ways$

3_ $C_1^3 \times C_3^9 = 252 \ ways$

ملاحظة:

1_ Less than $n \rightarrow x < n$ اقل من

2_ more than $n \rightarrow x > n$ اكثر من

3_ at must $n \rightarrow x \geq n$ على الأكثر

4_ at least $n \rightarrow x \leq n$ على الأقل

من المثال السابق ليكن لدينا المطاليب التالية :

4_ At least one boy ?

$C_1^9 \times C_3^3 + C_2^9 \times C_2^3 + C_3^9 \times C_1^3 + C_4^9 \times C_0^3$

5_ at must one girl ?

$C_1^3 \times C_3^9 + C_0^3 \times C_4^9$

ملاحظة مهمة : اذا كان لدينا a , b , c and d المطلوب اختيار ثلاثة حروف من الحروف الأربعة وفي حالتين وكما يلي :

الترتيب غير ضروري (غير مهم)	الترتيب ضروري (مهم)					
Combination	Permutation					
abc	abc	acb	bca	bac	cab	cba
abd	abd	adb	bda	bad	dab	dba
bcd	bcd	bdc	cdb	cbd	dbc	dcb
acd	acd	adc	cda	cad	dac	dca

$$C_3^4 = \frac{4!}{3!(4-3)!} = 4 \ ways \qquad P_3^4 = \frac{4!}{(4-3)!} = 24 \ ways$$

Exp) Assume that there are $n = 7$ students and that we wish to form 3 groups , 2 in the first , 3 in the sconed and 2 in the third .

Sol/ Let $n_1 = 2$, $n_2 = 3$ and $n_3 = 2$ then $n = n_1 + n_2 + n_3$

$$C_2^7 \times C_3^5 \times C_2^2 = \frac{7!}{2! * 5!} \times \frac{5!}{3! * 2!} \times \frac{2!}{2! * 0!} = \frac{7!}{2! * 3! * 2!}$$

In General :

$$C_{n_1}^n * C_{n_2}^{n-n_1} * C_{n_3}^{n-(n_1+n_2)} * \dots * C_{n_k}^{n-(n_1+n_2+n_3+\cdots+n_k)}$$

$$= \frac{n!}{n_1!\,(n-n_1)!} * \frac{(n-n_1)!}{n_2!\,(n-(n_1+n_2))!} * \dots * \frac{(n-(n_1+n_2+n_3+\cdots+n_{k-1}))!}{n_k!\,(n-(n_1+n_2+n_3+\cdots+n_k))!}$$

$$= \frac{n!}{n_1! * n_2! * n_3! * \dots * n_k!} \quad where \quad n = n_1 + n_2 + n_3 + \cdots + n_k$$

$$\therefore C_{n_1,n_2,n_3}^n = \frac{n!}{n_1! * n_2! * n_3!} = \frac{7!}{2! * 3! * 2!}$$

Exp) Prove the following :

1_ $C_r^n = C_{n-r}^n$

2- $C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$

3_ $C_r^{n+1} = C_{r-1}^n + C_r^n$

Proof /

1_ $C_r^n = C_{n-r}^n = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{r!(n-r)!}$

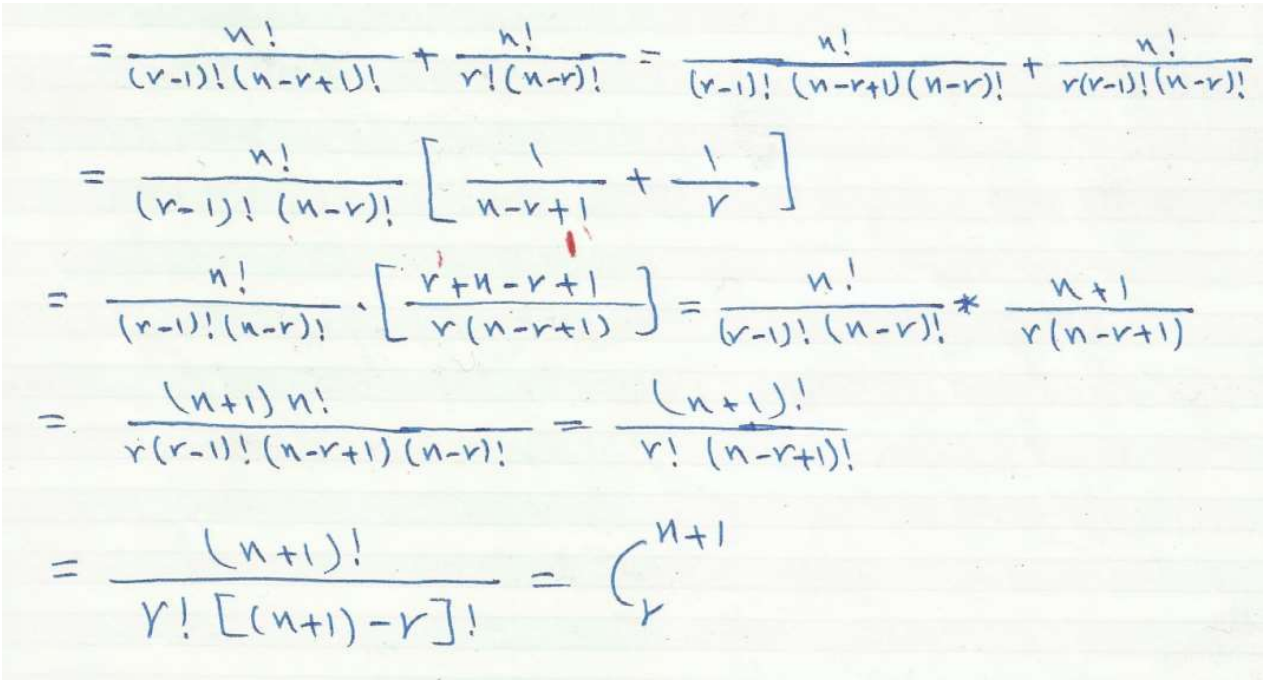
2_ $C_r^n = C_r^{n-1} + C_{r-1}^{n-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)![n-1-(r-1)]!}$

$= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} = \frac{(n-1)!}{r(r-1)!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)(n-r-1)!}$

$= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{1}{r} + \frac{1}{n-r}\right] = \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{n-r+r}{r(n-r)}\right] = \frac{(n-1)!}{(r-1)!(n-r-1)!} * \frac{n}{r(n-r)}$

$= \frac{n(n-1)!}{r(r-1)!(n-r)(n-r-1)!} = \frac{n!}{r!(n-r)!} = C_r^n$

3_ $C_r^{n+1} = C_{r-1}^n + C_r^n = \frac{n!}{(r-1)![n-(r-1)]!} + \frac{n!}{r!(n-r)!}$



2-4) Combinations and Binomial theorem

_ التوافيق ونظرية ثنائي الحدين _

If (n) is positive integer then :

$$(a+b)^n = \sum_{r=0}^{n} C_r^n\, a^r b^{n-r}$$

$$(a+b)^n = C_0^n a^0 b^n + C_1^n a^1 b^{n-1} + C_2^n a^2 b^{n-2} + \cdots + C_n^n a^n b^{n-n}$$

$$= b^n + nab^{n-1} + \frac{n(n-1)}{2!} a^2 b^{n-2} + \cdots + a^n$$

$\rightarrow n = 0 \;:\; (a+b)^0 = 1$

$n = 1 \;:\; (a+b)^1 = a+b$

$n = 2 \;:\; (a+b)^2 = a^2 + 2ab + b^2$

$n = 3 \;:\; (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$n = 4 \;:\; (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$n = 5 \;:\; (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$n = 6 \;:\; (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

مما تقدم يمكن ملاحظة الخواص التالية من مفكوك $(a+b)^n$

1_ يوجد في المفكوك $(n+1)$ من الحدود

2_ مجموع اسي كل من a و b يساوي n

3_ يتناقص اس a حدا بعد من n الى الصفر ويتزايد اس b من الصفر الى n حداً بعد حد .

4_ معامل كل حد هو C_r^n حيث ان r هو اس أي من a أو b .

5_ تتساوى معاملات الحدود التي تبتعد عن بداية المفكوك ونهايته بنفس المقدار .

6_ نلاحظ أيضاً ان معاملات القوى $(a+b)$ المتتالية يمكن ترتيبها في مثلث من الاعداد يسمى مثلث باسكال

n=0							**1**							
n=1							1		1					
n=2						1		2		1				
n=3					1		3		3		1			
n=4				1		4		6		4		1		
n=5			1		5		10		10		5		1	
n=6		1		6		15		20		15		6		1
n=7	1		7		21		35		35		21		7		1
n=8		1		8		28		56		70		56		28		8		1

$$\therefore\ (a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Lemma :-

For any positive integer n , we have :

$$(1+x)^n = \sum_{r=0}^{n} C_r^n\, x^r$$

Proof /

We have $(a+b)^n = \sum_{r=0}^{n} C_r^n\, a^r b^{n-r}$

Let $a = x$, $b = 1$

$$\rightarrow (1+x)^n = \sum_{r=0}^{n} C_r^n\, x^r 1^{n-r} = \sum_{r=0}^{n} C_r^n\, x^r$$

Prove that

$$2^n = \sum_{r=0}^{n} C_r^n$$

Proof / we have $(a+b)^n = \sum_{r=0}^{n} C_r^n\, a^r b^{n-r}$

Let $a = 1$ and $b = 1$

Then$\rightarrow (1+1)^n = \sum_{r=0}^{n} C_r^n\, 1^r 1^{n-r} \rightarrow 2^n = \sum_{r=0}^{n} C_r^n$

$$= C_0^n + C_1^n + C_2^n + \cdots + C_n^n$$

Exp) How many different subject can be made for a set A which contain 3 elements .

$$C_0^3 + C_1^3 + C_2^3 + C_3^3 = 1 + 3 + 3 + 1 = 8$$

Or

$$\rightarrow \ 2^n = 2^3 = 8$$

Exp) Prove that $(3x^2 + y)^5$

Proof/

We have $(a+b)^n = \sum_{r=0}^{n} C_r^n \, a^r b^{n-r}$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Let $a = 3x^2$ *and* $b = y$

$$(3x^2 + y)^5 = (3x^2)^5 + 5(3x^2)^4 y + 10(3x^2)^3 y^2 + 10(3x^2)^2 y^3 + 5(3x^2)y^4 + y^5$$

$$= 342x^{10} + 405x^8 y + 270x^6 y^2 + 90x^4 y^3 + 15x^2 y^4 + y^5$$

Exp) Prove that $(x + y)^5$

Proof/

We have $(a+b)^n = \sum_{r=0}^{n} C_r^n \, a^r b^{n-r}$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Let $a = x$ *and* $b = y$

$$\therefore (x+y)^5 = x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$$

2-5) Problems

1_ A coin is thrown and a dice is cast ; determine the number of sample points in the sample space of this experiment , If we tossing the coin first and if we tossing the dice first . (H.W)

2_ A dice is tossing three time , Determine the number of sample points in the sample space of this experiment . (H.W)

استخدم مخطط الشجرة لحل السؤال الأول والثاني .

3_ How many 3_digit numbers can be formed from the digits 2,4,6,7,9 ? (H.W)

4_ How many four number committee can be formed from a group of 7 person.(H.W)

5_A student is to answer 10 out of 13 questions on an exam.

I. How many choices has he? (H.W)
II. How many choices has if he must answer the first two questions?
III. How many choices has if he must answer the first or second questions?
IV. How many choices has if he must answer exactly 3 of the first 5 questions?
V. How many choices has if he must answer at least 3 of the first 5 questions?

6_ There are 5 roads between city A and city B ; and 4 roads between B and C .

I. In how many ways can one drive from city A to city C by way of B .
II. In how many ways can one drive from A to C and back to A passing through B on both trips without the same road more than once ?

Sol/

A B C

I. $5 * 4 = 20$
II. $5 * 4 * 3 * 4 = 240$

7_a_ In how many ways can a student arrange his 8 textbooks on a shelf ?

b_ In how many ways can he do this if three specific books must be together ?

Sol/

a_ $n! = 8! = 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 = 40320 \quad ways$

b_ [ABCDEFGH] على فرض ان الكتب تأخذ الاحرف التالية

$3! * 6! = 6 * 120 = 720$ ways

8_ Find n if i) $P_3^n = 336$ (H.W)

ii) $P_3^n = 5P_2^n$ (H.W)

iii) $2P_2^n + 50 = P_2^{2n}$

Sol/ $2P_2^n + 50 = P_2^{2n}$; we have $P_r^n = \frac{n!}{(n-r)!}$

$$\therefore 2\frac{n!}{(n-2)!} + 50 = \frac{2n!}{(2n-2)!} \rightarrow 2\frac{n(n-1)(n-2)!}{(n-2)!} + 50 = \frac{2n(2n-1)(2n-2)!}{(2n-2)!}$$

$$\rightarrow 2n(n-1) + 50 = 2n(2n-1)$$

$$\rightarrow 2n^2 - 2n + 50 = 4n^2 - 2n$$

$$\rightarrow 50 = 2n^2 \rightarrow n^2 = \frac{50}{2} = 25$$

$$\therefore n = \sqrt{25} = 5$$

9_ Mr. Ahmed has 5 suits , 7 ties and 4 pairs of shoes ; In how many ways can he chouse 3 suits , 4 ties and 2 pairs of shoes to take along on a business trip . (H.W)

Hint : $C_3^5 * C_4^7 * C_2^4$

10_ A student is to answer 7 questions out of 10 in an exam . How many choices has he :

I. If he must answer the first 3 questions
II. If he must not answer the last question ?

Hint : I. C_4^7 II. C_7^9

11_ If 4 Iraqis , 3 Egyptians and 2 Jordanians are to be seated in a row . In how many ways can this be done if the people of the same nationality must sit together ?

Sol/

$$n! = (4! * 3! * 2!) * 3! = 1728 \ ways$$

12_ Solve problem 11 if the sit at around table .

Sol/

$$(n-1)! = (4! * 3! * 2!) * 2! = 576 \ ways$$

13_ A class contains 10 boys and 5 girls : (H.W)

I. In how many ways can a teacher choose a committee of 4 ?
II. How many ways of these committee will contain 2 girls ?
III. How many of these committee will contain at least one boy ?

14_ A women has 11 class friends :

I. In how many ways can she invite 5 of them to dinner ?
II. In how many ways if two of the friends are married and will not attend separately ?
III. In how many ways if two of them are not on speaking terms and will not attend together ?

Sol/

I. C_5^{11}
II. $C_3^9 + C_5^9$
C_3^9 حضورهم معا $+ C_5^9$ في حالة عدم حضورهم
III. $C_5^{11} + C_4^9 + C_4^9$
C_5^{11} عدم حضورهم معا $+ C_4^9$ حضور الاول $+ C_4^9$ حضور الثاني

Chapter Three

Probability

3.1_ Introduction

Probability is the study of random or nondeterministic (غير محدد) experiment (that is ; we wish to consider some experiment results and the repetition of the experiment dose not always produce the same result).

For example , suppose one tossed a fair coin repeatedly , keeping a record of the number of heads h in the first n tossed ($n = 1,2,3, \dots$), then it has bean empirically (تجريبيا) observed (مشاهدة) that the ratio $f = h/n$, called the relative frequency becomes stable in the long run , i.e. approaches a limit (يقترب من الحد) . This stability is the basis of probability theory .

3.2_ Random Experiment , Sample Space and Events :

_ Definition :

_ Random Experiment : Is a procedure which results in some nondeterministic outcomes in a particular situation(حالة خاصة او معينة) .

_ Outcome : Is a single realization (مفهوم وحيد) of phenomenon under consideration (للظاهرة قيد البحث) , the outcome need not always be number or quantities which are representable in term of numbers .

_ Sample Space : The set of all possible outcomes of a random experiment denoted by S or Ω .

_ Events : It's a subset of sample space denoted by E .

_ Elementary Event (حدث ابتدائي او اولي) : Is an event consisting of only one element of the sample space S .

These points are illustrated (موضحة) with some examples :

1_ If the experiment consists of tossing a coin once , then

$$S = \{H, T\} \text{ where } H = head \text{ and } T = tail$$

When 2 tossing a coin :

$$S = \{HH, HT, TH, TT\}$$

When 3 tossing a coin :

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\} .$$

2_ If a couple is planning to have three children , then S consists of 8 elements ; i.e.

$$S = \{BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG\}$$

Where B means that the outcome of birth is boy and G is a girl .

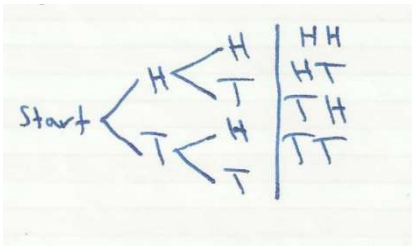
If E is the event that the couple will have 2 boys , then :

$$E = \{BBG, BGB, GBB\}$$

Also ; if F is the event that the couple will have at most on boy , then :

$$F = \{BGG, GBG, GGB, GGG\}$$

3_ Toss a two coins :



$$S = \{HH, HT, TH, TT\}$$

Let $A =$ one head appear

$\therefore\ A = \{HT, TH\}$

Let $B =$ no head appear or two tail appear

$\therefore\ B = \{TT\}$

Let $C =$ at must two head appear

$\therefore\ C = \{HH, HT, TH\}$

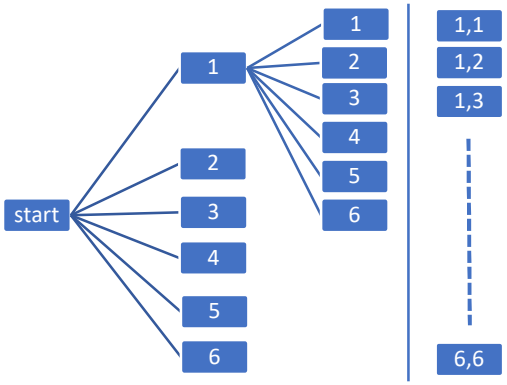
Let $D =$ at least two head appear

$\therefore\ D = \{HH\}$

4_ If the experiment consists of tossing a pair of dice, then the sample space consists of 36 elements; i.e.

$$S = \begin{bmatrix} (1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6) \\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6) \\ (3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6) \\ (4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6) \\ (5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6) \\ (6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6) \end{bmatrix}$$

or



Define the following events as :

$A = \{the\ sum\ on\ the\ two\ dice\ is\ 6\ \}$

$B = \{both\ dice\ show\ the\ same\ number\ \}$

$C = \{at\ least\ one\ of\ them\ is\ 4\ \}$

Then :

$A = \{(1,5)\ ;\ (2,4)\ ;(3,3)\ ;(4,2)\ ;(5,1)\}$

$B = \{(1,1)\ ;\ (2,2)\ ;(3,3)\ ;(4,4)\ ;(5,5)\ ;(6,6)\}$

$C = \{(1,4)\ ;\ (2,4)\ ;(3,4)\ ;(4,4)\ ;(5,4)\ ;(6,4)\ ;\ (4,1)\ ;\ (4,2)\ ;(4,3)\ ;(4,5)\ ;(4,6)\}$

5_ Consider the experiment involving the toss of single dice ; then :

$S = \{1,2,3,4,5,6\}$

$A = \{be\ an\ event\ of\ odd\ number\ \}$

$B = \{be\ an\ event\ of\ even\ number\ \}$

$C = \{divisible\ by\ 3\ \}$

$D = \{less\ than\ 4\ \}$

Then :

$A = \{1,3,5\}$; $B = \{2,4,6\}$

$C = \{3,6\}$; $D = \{1,2,3\}$

Remarks :

1_ $A \cup B$: Is a subset of sample space contain the element in A or B or Both

2_ $A \cap B$: Is a subset of sample space contain the element in A and B

3_ A^c : Is a subset of sample space contain the element in S but not in A

4_ A/B : The event (A) appear and event (B) not appear

5_ $A \cap B = \emptyset$: Event (A) and (B) are mutually exclusive (m.e.)

3.3 _ Kinds of Probability أنواع الاحتمال

There are three kinds of probability and these are :

1_ The Classical Approach . الطريقة الكلاسيكية

2_ The relative Frequency Approach . طريقة التردد النسبي

3_ The Subjective Approach . طريقة النظرة الشخصية

سوف نتعرف على الطريقة الكلاسيكية (التقليدية) ويعتبر القانون الأول في الاحتمالية :

3.3.1_ The Classical Approach :

Suppose an event E can happen in h ways out of total of n possible equally likely ways . Then the probability of occurrence (ظهور) of the event { called its Success} is denoted by :

$$P = P(E) = \frac{h}{n}$$

الجزء

الكل

$$0r = \frac{n(E)}{n}$$

That's mean : the probability of event E equals the number of outcomes favorable to E divided by the total number of outcomes .

$$\text{i.e. : } P(E) = \frac{favorable\ outcomes}{total\ outcomes}$$

The Probability of non-occurrence of the event , { called its failure } is denoted by :

$$q = P(not\ E) = \frac{n-h}{n}$$

$$or\ q = 1 - \frac{h}{n} = 1 - P = 1 - P(E)$$

Thus :

$$P + q = 1$$

or

$$P(E) + P(not\ E) = 1$$

Exp)

1_ A fair coin is tossed once , then $S = \{H, T\}$:

Thus, the probability of getting a head is $P = \frac{1}{2}$

and the probability of getting a tail is $P = \frac{1}{2}$

2_ When an unbiased dice is thrown once , then $S = \{1,2,3,4,5,6\}$

The six outcomes are mutually exclusive because two or more faces cannot turn up simultaneously (يظهر في وقت واحد) then :

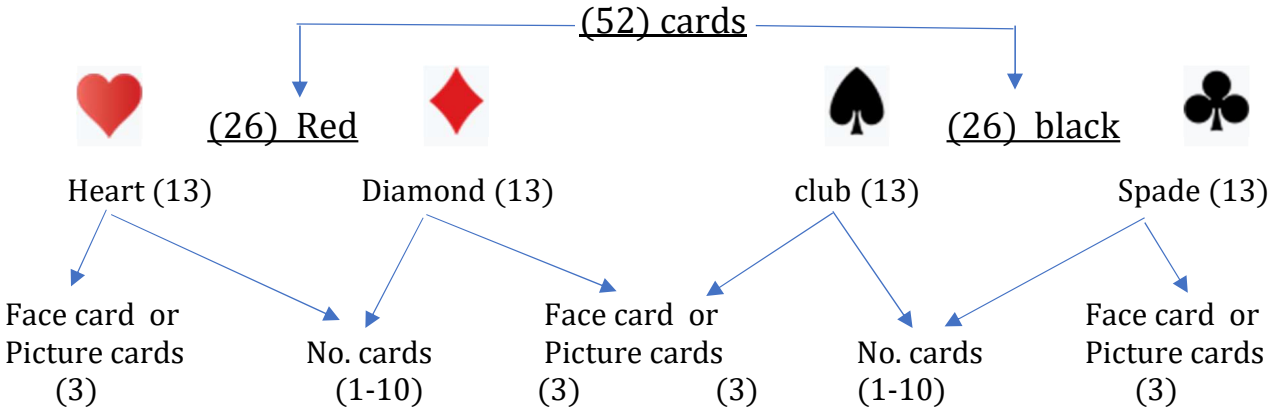
$$P = (1) = P = (2) = P = (3) = P = (4) = P = (5) = P = (6) = \frac{1}{6}$$

To fined the probability of getting an odd number , define

$A = \{1,3,5\}$

$$\rightarrow P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{3}{6} = \frac{1}{2}$$

3_ What is the probability of getting a cards :



Number Cards (40)		Faced or Picture cards (12)
Diamond ♦ :	(1-10)	Jake ; Queen ; King
Heart ♥ :	(1-10)	Jake ; Queen ; King
Club ♠ :	(1-10)	Jake ; Queen ; King
Spade ♣ :	(1-10)	Jake ; Queen ; King

1_ one Picture ; $P(A) = \frac{h}{n} = \frac{12}{52}$ or $\frac{C_1^{12}}{C_1^{52}}$

2_ one Number ; $P(B) = \frac{h}{n} = \frac{40}{52}$ or $\frac{C_1^{40}}{C_1^{52}}$

From the classical definition of probability of an event A we have :

1_ $0 \leq P(A) \leq 1$; $0 \leq \frac{h}{n}$ or $\frac{n(A)}{n} \leq 1$ for every n .

2_ If A is a certain event , then $P(A) = 1$ because $h = n$ 0r $n(A) = n$ and if A is an impossible event (null event); then :

$P(A) = 0$ because $h = n(A) = 0$

مثل رمي زهر النرد والمطلوب احتمال ظهور رقم اكبر من 6 .

3_ If A and B are two mutually exclusive events , then :

$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$ because

$P(A \cup B) = \frac{h}{n} = \frac{n(A \cup B)}{n} = \frac{n(A)+n(B)}{n} = \frac{n(A)}{n} + \frac{n(B)}{n} = P(A) + P(B)$

Where $n(A), n(B)$ and $n(A \cup B)$ are the number of occurrences of the events A, B and $(A \text{ or } B)$ respectively .

The result can be extended to the case of K mutually exclusive (m.e.) events $A_1, A_2, A_3, A_4, \ldots, A_k$ i.e.

$P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } \ldots \text{or } A_k = P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k) =$

$P(\bigcup_{i=1}^{k} A_i) = \sum_{i=1}^{k} P(A_i)$

Exp) Let a card be selected at random from an ordinary deck of 52 cards ;

$A = \{ \text{the card is spade} \}$ and $B = \{ \text{the card is a faced card} \}$, i.e. a Jake , Queen or King ; Compute :

$P(A)$, $P(B)$ and $P(A \cap B)$

Sol/

Since we have an equiprobable space

$$P(A) = \frac{number\ of\ spades}{number\ of\ cards} = \frac{h}{n} = \frac{n(A)}{n} = \frac{C_1^{13}}{C_1^{52}} = \frac{13}{52}$$

$$P(B) = \frac{number\ of\ face\ cards}{number\ of\ cards} = \frac{h}{n} = \frac{n(B)}{n} = \frac{C_1^{12}}{C_1^{52}} = \frac{12}{52}$$

$$P(A \cap B) = \frac{number\ of\ spade\ face\ cards}{number\ of\ cards} = \frac{h}{n} = \frac{n(A \cap B)}{n}$$
$$= \frac{C_1^3}{C_1^{52}} = \frac{3}{52}$$

Exp) Let 2 items be chosen at random from vessel (وعاء) containing 12 items of which 4 are defective .

Let $A = \{ both\ items\ are\ defective \}$

Let $B = \{ both\ items\ are\ non - defective \}$

Find $P(A)$ and $P(B)$.

Sol/

$$P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{n(A)}{S} = \frac{C_2^4}{C_2^{12}} = \frac{\frac{4!}{2!*2!}}{\frac{12!}{2!*10!}}$$

$$\therefore P(A) = \frac{6}{66} = \frac{1}{11}$$

$$P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{n(B)}{S} = \frac{C_2^8}{C_2^{12}} = \frac{\frac{8!}{2!*6!}}{\frac{12!}{2!*10!}}$$

$$\therefore P(A) = \frac{28}{66} = \frac{14}{33}$$

Let $C = \{ one\ is\ defective\ and\ one\ is\ non-defective \}$

$$\therefore P(C) = \frac{h}{n} = \frac{n(C)}{n} = \frac{C_1^4 * C_1^8}{C_2^{12}} = \frac{4*8}{66} = \frac{32}{66} = \frac{16}{33}$$

Exp)

Two digits are chosen at random from the digits {1,2,3,4,5} ; List the 2-digits numbers that can be formed . What is the probability that the number is even?

Sol/

$$\underline{5}\ \underline{4} = 5 \times 4 = 20 \text{ ways or}$$

$$P_r^n = P_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 5 * 4 = 20 \text{ ways}$$

{12 , 13 , 14 , 15 , 21 , 23 , 24 , 25 , 31 , 32 , 34 , 35 , 41 , 42 , 43 , 45 , 51 , 52 , 53 , 54}

Even number :

$$\underline{4}\ \underline{2} = 4 \times 2 = 8 \text{ ways}$$

$$\therefore P(even) = \frac{h}{n} = \frac{8}{20} = 0.4$$

Exp) A Committee of 4 persons are chosen at random from a class containing 7 girls and 9 boys . What is the probability that the committee will contain :

1_ 3 boys .

2_ at least one girl .

Sol/

$$S = C_4^{16} = \frac{n!}{r!(n-r)!} = \frac{16!}{4!*12!} = 1820$$

1_ 3 boys : $C_3^9 C_1^7 = \frac{9!}{3!*6!} * \frac{7!}{1!*6!} = 84 * 7 = 588$

$$\therefore P(3\ boys) = \frac{C_3^9 C_1^7}{C_4^{16}} = \frac{588}{1820} = 0.3231$$

2_ $B = \{at\ least\ one\ girl\}$

$$\therefore B = C_3^9 C_1^7 + C_2^9 C_2^7 + C_1^9 C_3^7 + C_0^9 C_4^7$$

$$= \frac{9!}{3!*6!} * \frac{7!}{1!*6!} + \frac{9!}{2!*7!} * \frac{7!}{2!*5!} + \frac{9!}{1!*8!} * \frac{7!}{3!*4!} + \frac{9!}{0!*9!} * \frac{7!}{4!*3!}$$

$$= 84 * 7 + 36 * 21 + 9 * 35 + 1 * 35 = 1694$$

$$\therefore P(B) = \frac{C_3^9 C_1^7 + C_2^9 C_2^7 + C_1^9 C_3^7 + C_0^9 C_4^7}{C_4^{16}} = \frac{1694}{1820} = 0.931$$

Or

$$P(B) = 1 - P(not\ girl) = 1 - \frac{C_4^9 C_0^7}{C_4^{16}} = 1 - \frac{126}{1820} = 1 - 0.06923 = 0.931$$

3_ Let $D = \{exactly\ 3\ girl\}$

$$\therefore P(D) = \frac{n(D)}{n} = \frac{C_3^7 C_1^9}{C_4^{16}} = \frac{35 * 9}{1820} = \frac{315}{1820} = 0.1731$$

4_ Let $F = \{at\ least\ one\ boy\}$

$$\therefore P(F) = \frac{h}{n} = \frac{n(F)}{n} = \frac{C_1^9 C_3^7 + C_2^9 C_2^7 + C_3^9 C_1^7 + C_4^9 C_0^7}{C_4^{16}} = 0.981$$

Or

$$P(F) = 1 - P(not\ boy) = 1 - \frac{C_0^9 C_4^7}{C_4^{16}} = 1 - \frac{35}{1820} = 1 - 0.0192 = 0.9808$$

Exp)

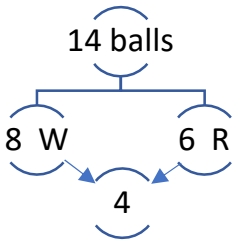
A bag contains 8 white and 6 red balls , 4 balls are drawn at random from this bag ; Finds the probability of the following events :

A = { there will be 2 red balls }

B = { there will be at least 2 white balls }

D = { at most 2 red balls }

Sol/



$$S = C_4^{14} = \frac{n!}{r!(n-r)!} = 1001$$

$$A = C_2^6 C_2^8 = \frac{6!}{2!*4!} * \frac{8!}{2!*6!} = 15 * 28 = 420$$

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{420}{1001} = 0.4196$$

$$B = C_2^8 C_2^6 + C_3^8 C_1^6 + C_4^8 C_0^6$$

$$= \frac{8!}{2!*6!} * \frac{6!}{2!*4!} + \frac{8!}{3!*5!} * \frac{6!}{1!*5!} + \frac{8!}{4!*4!} * \frac{6!}{0!*6!}$$

$$= 28 * 15 + 56 * 6 + 70 * 1 = 826$$

$$\therefore P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{826}{1001} = 0.8252$$

$$D = C_2^6 C_2^8 + C_1^6 C_3^8 + C_0^6 C_4^8$$

$$\therefore P(D) = \frac{C_2^6 C_2^8 + C_1^6 C_3^8 + C_0^6 C_4^8}{C_4^{14}} = \frac{15*28 + 6*56 \quad 1*70}{1001} = \frac{826}{1001} = 0.8252$$

Exp)

Three cards are drawn at random from a deck of 52 cards, let the events A, B, C, D and E be defined as:

$A = an\ ace$

$B = 2\ cards\ of\ daimond$

$C = 2\ picture\ cards\ and\ one\ numberd\ card\ of\ spad$

$D = one\ numbered\ card, one\ Jack\ and\ the\ Queen\ of\ heart$

$E = the\ first\ card\ is\ an\ ace\ and\ the\ others\ one\ picture\ cards$

Find the probability of the above events.

Sol/

$$S = C_3^{52} = \frac{52!}{3!*49!} = 22100$$

$$A = C_1^4 C_2^{48} = \frac{4!}{1!*3!} * \frac{48!}{2!*46!} = 4 * 1128 = 4512$$

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{4512}{22100} = 0.20416$$

$$B = C_2^{13} C_1^{39} = \frac{13!}{2!*11!} * \frac{39!}{1!*38!} = 78 * 39 = 3042$$

$$\therefore P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{3042}{22100} = 0.13765$$

$$C = C_1^{10} C_2^{12} = \frac{10!}{1!*9!} * \frac{12!}{2!*10!} = 10 * 66 = 660$$

$$\therefore P(C) = \frac{h}{n} = \frac{n(C)}{n} = \frac{660}{22100} = 0.002986$$

$$D = C_1^{40} C_1^4 C_1^1 = \frac{40!}{1!*39!} * \frac{4!}{1!*3!} * \frac{1!}{1!*0!} = 40 * 4 * 1 = 160$$

$$\therefore P(D) = \frac{h}{n} = \frac{n(D)}{n} = \frac{160}{22100} = 0.00224$$

ملاحظة مهمة : عندما يذكر ترتيب معين لعملية الاختيار كما هو الحال في الحدث (E) حيث ذكر ان الورقة الأولى التي سحبت هي من (الآس " الواحد في ورق اللعب") والورقتان الثانية والثالثة من الصور ، عندئذٍ نستخدم في الحل التباديل بدلاً من التوافيق :

$$E = P_1^4 * P_2^{12}$$

$$\therefore P(E) = \frac{P_1^4 * P_2^{12}}{P_3^{52}} = \frac{\frac{4!}{3!} * \frac{12!}{10!}}{\frac{52!}{49!}} = \frac{528}{22100} = 0.003982$$

Exp)

Three light bulbs are chosen at random from 15 bulbs which are 5 of them are defective ; Find the probability that :

A = {non defective}

B = {exactly one is defective}

C = {at least one is defective}

Sol/

$$S = C_3^{15} = \frac{15!}{3!*12!} = 455$$

$$A = C_3^{10} = \frac{10!}{3!*7!} = 120$$

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{120}{455} = 0.2637$$

$$B = C_1^5 * C_2^{10} = \frac{5!}{1!*4!} * \frac{10!}{2!*8!} = 5 * 45 = 225$$

$$\therefore P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{225}{455} = 0.4945$$

$$C = C_1^5 * C_2^{10} + C_2^5 * C_1^{10} + C_3^5 * C_0^{10} =$$

$$= \frac{5!}{1!*4!} * \frac{10!}{2!*8!} + \frac{5!}{2!*3!} * \frac{10!}{1!*9!} + \frac{5!}{3!*1!} * \frac{10!}{0!*10!} = 335$$

$$\therefore P(C) = \frac{h}{n} = \frac{n(C)}{n} = \frac{335}{455} = 0.7363$$

Or

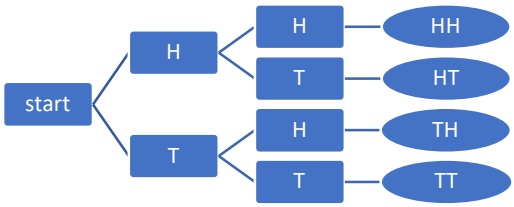
$$P(C) = 1 - P(A) = 1 - \frac{C_3^{10} * C_0^5}{C_3^{15}} = 1 - 0.2637 = 0.7363$$

Exp)

Two coins are to be tossed once , Find the possible outcomes of this experiments ?

Sol/

H : head ; T : tail



$\therefore S = \{HH, HT, TH, TT\}$

Exp)

A pair of dice is to be rolled once, Find the following events?

A _ The event that the sum of the two faces is 7.

B _ The event that the sum of the two faces is 11.

C _ The event that the sum of the two faces is either 7 or 11.

Sol/

A= {(1,6) ;(2,5) ;(3,4) ;(4,3) ;(5,2) ;(6,1)}

B= {(5,6) ;(6,5)}

C= $A \cup B$ = {(1,6) ;(2,5) ;(3,4) ;(4,3) ;(5,2) ;(6,1) ;(5,6) ;(6,5)}

لو كان المطلوب هو إيجاد probability of these events

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{6}{36}$$

$$\therefore P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{2}{36}$$

$$\therefore P(C) = \frac{h}{n} = \frac{n(C)}{n} = \frac{8}{36}$$

$E_1 = \{ The\ first\ face\ is\ greater\ than\ second\ face \}$

$E_1 =$ {(2,1); (3,1); (3,2); (4,1); (4,2); (4,3); (5,1); (5,2); (5,3); (5,4);
(6,1); (6,2); (6,3); (6,4); (6,5) }

$$\therefore P(E_1) = \frac{h}{n} = \frac{n(E_1)}{n} = \frac{15}{36}$$

$E_2 = \{ The\ difference\ between\ two\ faces\ zero \}$

$E_2 =$ {(1,1) ; (2,2) ; (3,3) ; (4,4) ; (5,5) ; (6,6)}

$$\therefore P(E_2) = \frac{h}{n} = \frac{n(E_2)}{n} = \frac{6}{36}$$

$E_3 = \{ The\ second\ face\ shows\ even\ number \}$

$E_4 = \{ The\ first\ face\ is\ an\ even\ and\ the\ second\ is\ 5 \}$

$E_5 = \{ The\ second\ face\ shows\ a\ number\ 3\}$

$E_6 = \{ The\ second\ face\ shows\ number\ 1\ or\ 5\ and\ the\ sum\ of$
$two\ faces\ (number) is\ less\ than\ 8\ and\ more\ than\ 3\ \}$

_____(H.W)

Chapter Four

_ Axiomatic Approach of Probability _

4.1_ Probability Defined on Event: -

In this chapter we develop mathematical model of an experiment which is defined mathematically by three things.

1_ Specifying the sample space S on which probability statements are made;

تحديد فضاء العينة التي يتم فيها وضع بيانات الاحتمالية؛

2_ defining the events of interest; and

تحديد الأحداث ذات الأهمية؛ و

3_ defining a numerical measure for probability statements, which is assigned to the events.

تحديد مقياس رقمي للبيانات الاحتمالية، والذي يتم تعيينه للأحداث

To each event A defined on a sample space S we shall assign a non-negative real number which is called probability of A , denoted by $P(A)$.

4.1_ Axioms of Probability: بديهيات الاحتمال

Let S be sample space, let σ be the class of events, and let P be a real value function defined on σ .

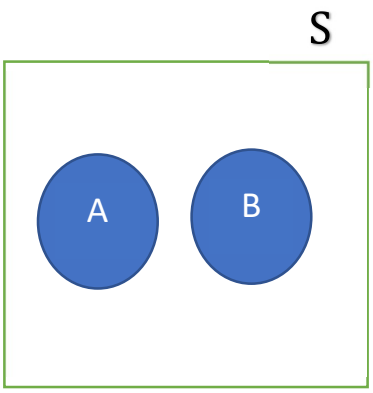
Then P is called a probability function, and $P(A)$ is called probability of the event A if the following axioms hold

1_ For every event A , $P(A) \geq 0 \quad ; \quad 0 \leq P(A) \leq 1$

2_ $P(S) = 1 \quad ; \quad P(S) = \frac{h}{n} = \frac{n(S)}{n(S)} = 1$

3_ If $A \quad and \quad B$ are mutually exclusive events, then:

$$P(A \cup B) = P(A) + P(B)$$

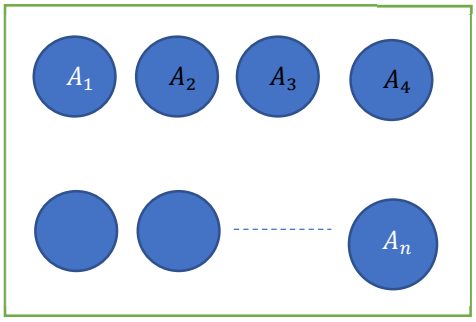


4_ If A_1 , A_2 , A_3 , … , A_n is a sequence of mutually exclusive events, then:

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(\bigcup_{i\in I} A_i) = \sum_{i=1}^{n} P(A_i)$$

$$= P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$$

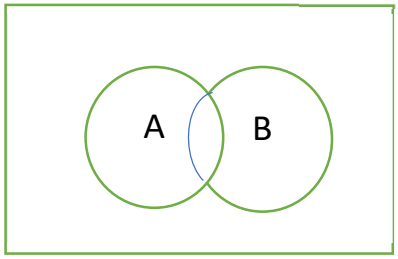
S



5_ If A *and* B are not mutually exclusive events, then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

S



Theorem (1) :

$P(\emptyset) = 0$ for any sample space.

Proof:

Let A be any subset , then A and $\emptyset$ are disjoint , and

$A = A \cup \emptyset \quad \rightarrow \quad P(A) = P(A \cup \emptyset)$

Since A and $\emptyset$ are **m.e.** events by axiom (3)

$\rightarrow P(A \cup \emptyset) = P(A) + P(\emptyset)$

$\rightarrow P(A) = P(A) + P(\emptyset)$

$\rightarrow P(\emptyset) = 0$

Or

$$P(\emptyset) = \frac{n(\emptyset)}{n(S)} = \frac{0}{n} = 0$$

Theorem (2) :

For any event $A \subset S$, $P(A^c) = 1 - P(A)$

Proof

Let $S = A \cup A^c \quad \rightarrow \quad P(S) = P(A \cup A^c)$

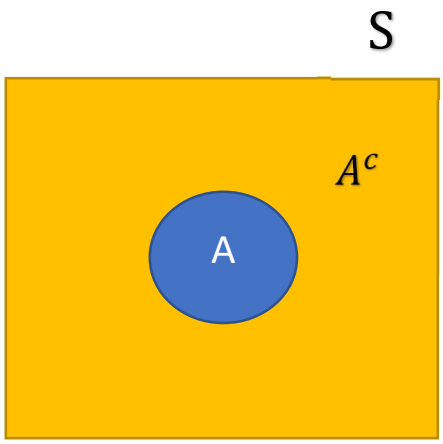
Since A & A^c are m.e. events, then:

$P(A \cup A^c) = P(A) + P(A^c)$

$\therefore \; P(S) = 1$

$\therefore \quad 1 = P(A) + P(A^c)$

$\therefore \quad P(A) = 1 - P(A^c)$



Or

$$P(A^c) = 1 - P(A)$$

$\therefore \quad A \cap A^c = \emptyset$

&

$$P(A \cap A^c) = P(\emptyset) = 0$$

Theorem (3) :

$$0 \leq P(A) \leq 1$$

Proof

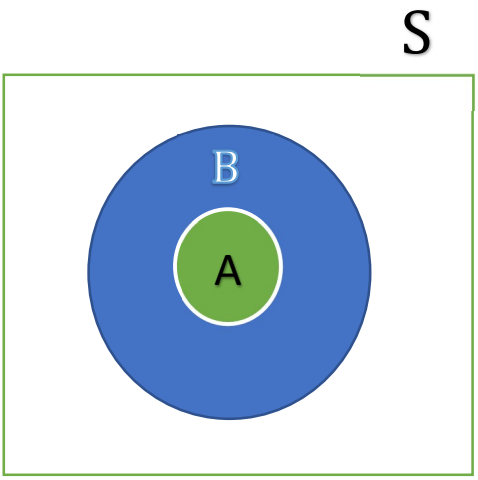
$\because \; P(\emptyset) = 0$

& $P(S) = 1$, $P(\emptyset) \leq P(A) \leq P(S)$

$\therefore \quad 0 \leq P(A) \leq 1$

Theorem (4) :

If $A \subset B$, *then* $P(A) \leq P(B)$



Proof

We have $B = A \cup (B \cap A^c)$ (m.e.)

$\therefore \quad P(B) = P[A \cup (B \cap A^c)]$

$\rightarrow \quad P(B) = P(A) + P(B \cap A^c)]$

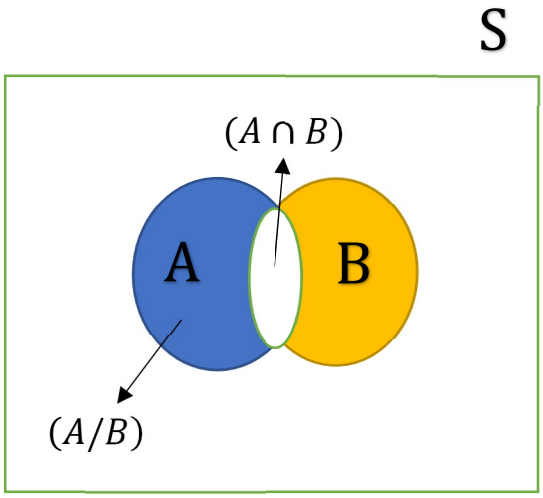
كمية موجبة

$\therefore \quad P(B) \geq P(A)$

Theorem (5) :

For any two events A *and* B

$P(A/B) = P(A) - P(A \cap B)$



Proof

Let $A = (A/B) \cup (A \cap B)$

$\rightarrow P(A) = P[(A/B) \cup (A \cap B)]$ (m.e)

$= P(A/B) + P(A \cap B)$

$\rightarrow P(A/B) = P(A) - P(A \cap B)$

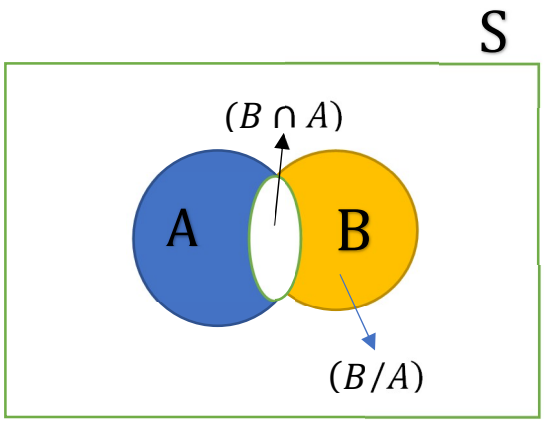
ويمكن *كتابة* A/B بالشكل التالي

$A/B = A \cap B^c$

وكذلك يمكن كتابة B/A بالشكل التالي

$B/A = B \cap A^c$

$\&\ P(B/A) = P(B) - P(B \cap A)$



Proof

Let $B = (B/A) \cup (B \cap A)$

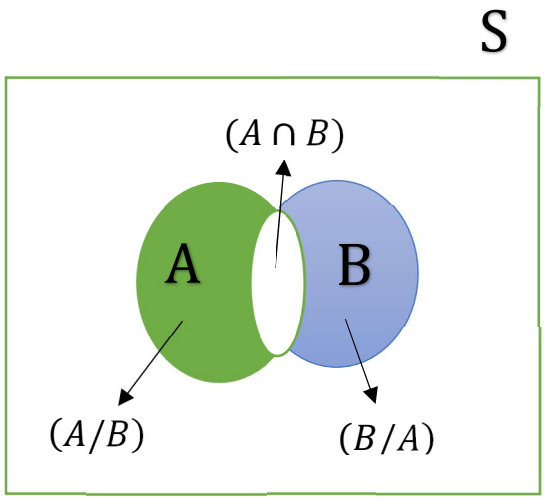
$\rightarrow P(B) = P[(B/A) \cup (B \cap A)]$

m.e

$= P(B/A) + P(B \cap A)$

$\therefore\ P(B/A) = P(B) - P(B \cap A)$

Theorem (6) :



For any two events A *and* B

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof

Let $A \cup B = A \cup (B/A)$

$\rightarrow P(A \cup B) = P[(A) \cup (B/A)]$

m.e

$= P(A) + P(B/A)$

$\because\ P(B/A) = P(B) - P(B \cap A)$

$\therefore\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof with another way:

$(A \cup B) = (A/B) \cup (A \cap B) \cup (B/A)$

m.e

$\therefore P(A \cup B) = P[(A/B) \cup (A \cap B) \cup (B/A)]$

$= P(A/B) + P(A \cap B) + P(B/A)$

$\because\ P(A/B) = P(A) - P(A \cap B)$

$\&\ P(B/A) = P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) + P(B) - P(A \cap B)$

Proof again: -

$(A \cup B) = (A \cap B^c)\ \cup (A \cap B) \cup (B \cap A^c)$

m.e

$\therefore P(A \cup B) = P[(A \cap B^c) \cup (A \cap B) \cup (B \cap A^c)]$

$= P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$

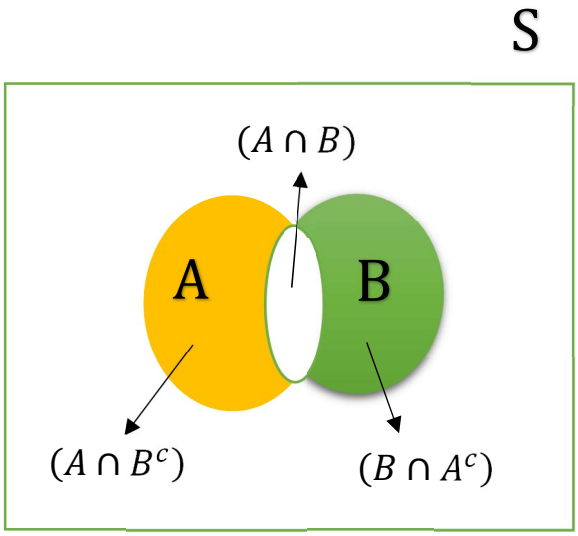
$\because\ (A/B)\ =\ (A \cap B^c)$

$\therefore\ P(A \cap B^c) = P(A) - P(A \cap B)$

$\&\ P(B \cap A^c) = P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) + P(B) - P(A \cap B)$

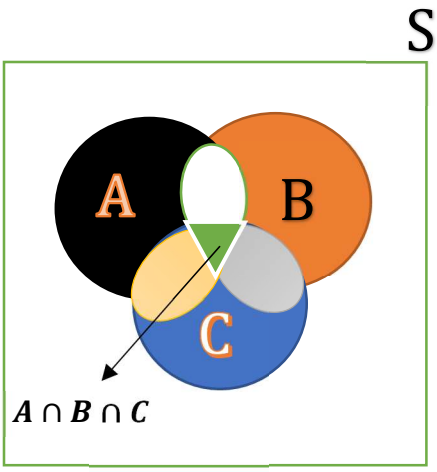


Corollary: -

If A , B & C are any three events; then:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof:



$$P(A \cup B \cup C) = P[A \cup (B \cup C)]$$

$$\therefore P[A \cup (B \cup C)] = P(A) + P(B \cup C) - P[A \cap (B \cup C)]$$

And we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\therefore P[A \cup (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

And $P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

4_2_1_ Examples of probability laws:

EX/

Two coins are tossed, what is the probability that at least one head appears?

SOL/

$$S = \{HH, HT, TH, TT\}$$

$$A = \{at\ least\ one\ head\ appears\} = \{HH, HT, TH\} = 3$$

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{3}{4} \quad or \quad \frac{C_1^3}{C_1^4}$$

$$B = \{exactly\ one\ head\ appear\} = \{HT, TH\} = 2$$

$$\therefore P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{2}{4} \quad or \quad \frac{C_1^2}{C_1^4}$$

$$C = \{exactly\ two\ head\ appear\} = \{HH\} = 1$$

$$\therefore P(C) = \frac{h}{n} = \frac{n(C)}{n} = \frac{1}{4} \quad or \quad \frac{C_1^1}{C_1^4}$$

$$(A \cap B) = \{HT, TH\} = 2$$

$$\therefore P(A \cap B) = \frac{h}{n} = \frac{n(A \cap B)}{n} = \frac{2}{4} \quad or \quad \frac{C_1^2}{C_1^4}$$

EX/

Two dice are thrown once, let A & B be two events defined by:

$A = \{$ *the first dice shows the number* $1\}$

$B = \{$ *the sum of the two numbers appearing is less than* $6\}$

Find $P(A)$, $P(B)$, $P(A \cup B)$ & $P(A \cup B^c)$

SoL/

$A = \{ (1,1); (1,2); (1,3); (1,4); (1,5); (1,6)\}$

$B = \{ (1,1); (1,2); (2,1); (1,3); (3,1); (1,4); (4,1); (2,2); (2,3); (3,2)\}$

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{6}{36} \quad or \quad \frac{C_1^6}{C_1^{36}}$$

$$\therefore P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{10}{36} \quad or \quad \frac{C_1^{10}}{C_1^{36}}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore (A \cap B) == \{ (1,1); (1,2); (1,3); (1,4)\}$$

$$\therefore P(A \cap B) = \frac{h}{n} = \frac{n(A \cap B)}{n} = \frac{4}{36}$$

$$\therefore P(A \cup B) = \frac{6}{36} + \frac{10}{36} - \frac{4}{36} = \frac{12}{36}$$

$$A^c = \{(2,1);(2,2);(2,3);\dots;(6,4);(6,5);(6,6)\}$$

$$\therefore P(A^c) = \frac{h}{n} = \frac{n(A^c)}{n} = \frac{30}{36} \quad or \quad P(A^c) = 1 - P(A) = 1 - \frac{6}{36} = \frac{30}{36}$$

$$\therefore P(B^c) = 1 - P(B) = 1 - \frac{10}{36} = \frac{26}{36}$$

$$\therefore P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c)$$

And we have

$$(A \cap B^c) = (A/B)$$

$$\therefore P(A \cap B^c) = P(A/B) = P(A) - P(A \cap B)$$

$$= \frac{6}{36} - \frac{4}{36} = \frac{2}{36}$$

$$\therefore P(A \cup B^c) = \frac{6}{36} + \frac{26}{36} - \frac{2}{36} = \frac{30}{36}$$

$$P(A^c \cup B) = H.W.$$

$$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c) \qquad \text{.....H.W}$$

Where $(A^c \cap B^c) = (A \cap B)^c$

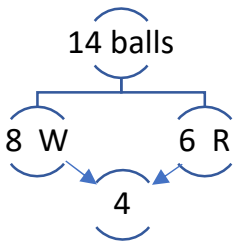
EX/

A bag contains 8 white and 6 red balls ; 4 balls are drawn at random from this bag ; Find the probability of the following events?

A= {There will be 2 red balls} .

B= {There will be at least 2 white balls} .

Find $P(A \cup B)$



SoL/

$$S = C_4^{14} = \frac{14!}{4! * (10)!} = 1001$$

$$A = C_2^6 C_2^8 = \frac{6!}{2!*4!} \times \frac{8!}{2!*6!} = 15 \times 28 = 420$$

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{420}{1001} = 0.4196$$

$$P(B) = \frac{C_2^8 C_2^6 + C_3^8 C_1^6 + C_4^8 C_0^6}{C_4^{14}} = 0.8252$$

$$P(A \cap B) = \frac{C_2^8 C_2^6}{C_4^{14}} = 0.4196$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.4196 + 0.8252 - 0.4196 = 0.8252$$

Example – A class contains 10 men and 20 women of which half the men and half the women have brown eyes, find the probability that a person chosen at random is a man or has brown eyes ?

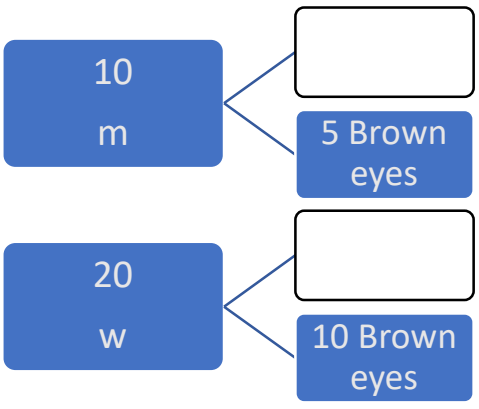
Solution/

let A = person is a man

& B = person has brown eyes

we seek $P(A \cup B)$

Then :



$$P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{10}{30}$$

$$P(B) = \frac{h}{n} = \frac{n(B)}{n} = \frac{15}{30}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n} = \frac{5}{30}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{10}{30} + \frac{15}{30} - \frac{5}{30} = \frac{20}{30} = \frac{2}{3}$$

H.W.

From the above example find the probability that a person chosen at random is a woman or with non-brown eyes

A = a person is a woman

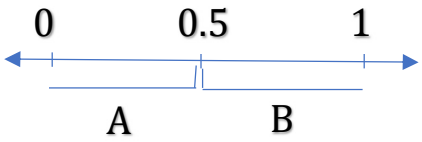
B = a person with non-brown eyes

Hint / we also seek about $\boldsymbol{P(A \cup B)}$

Problems [Axiomatic Approach of Probability]

1- Let the sample space $\boldsymbol{S = \{x: 0 < x < 1\}}$ $\boldsymbol{if\ A = \{x: 0 < x < 0.5\}}$ $\boldsymbol{and}$ $\boldsymbol{B = \{x: 0.5 \leq x < 1\}}$, Find $\boldsymbol{P(B)}$ $\boldsymbol{if}$ $\boldsymbol{P(A) = 1/4}$

SoL/ $\boldsymbol{A}$ & $\boldsymbol{B}$ are m.e. events



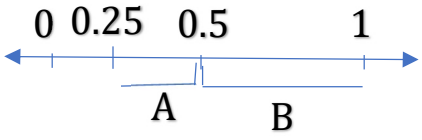
The $\boldsymbol{P(A \cup B) = P(A) + P(B)}$

$$\boldsymbol{1 = \frac{1}{4} + P(B)}$$

$$\boldsymbol{\therefore\ P(B) = 1 - \frac{1}{4} = \frac{3}{4}}$$

H.W. $\boldsymbol{P(A^c)}$, $\boldsymbol{P(B^c)}$, $\boldsymbol{P(A \cap B^c)}$ & $\boldsymbol{P(A^c \cup B^c)}$

2- Let the subset $\boldsymbol{A = \{x: 0.25 < x < 0.5\}}$ & $\boldsymbol{B = \{x: 0.5 \leq x < 1\}}$ the sample space $\boldsymbol{S = \{x: 0 < x < 1\}}$ be such that $\boldsymbol{P(A) = \frac{1}{8}}$ $\boldsymbol{and}$ $\boldsymbol{P(B) = \frac{1}{2}}$, Find $\boldsymbol{P(A \cap B)}$, $\boldsymbol{P(A \cup B)}$, $\boldsymbol{P(A^c)}$ & $\boldsymbol{P(A^c \cap B^c)}$ ***H.W.***



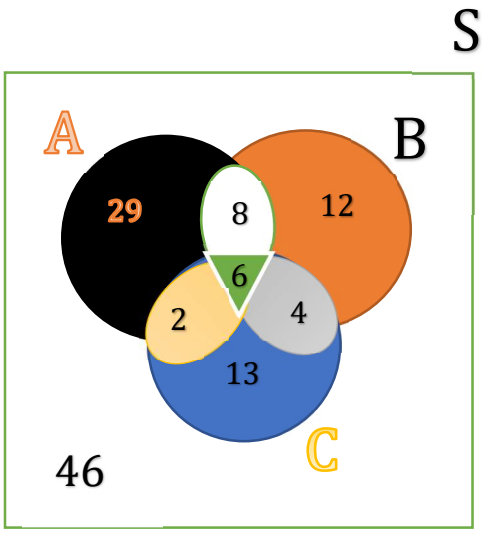
Example : the Venn diagram be of 120 students, 45 are studying French, 30 are studying Spanish, 25 are studying Italian, 14 are studying French and Spanish; 10 are studying Spanish and Italian; 8 are studying French and Italian; and 6 are studying the three languages. If a student is chosen at random, find the probability that the student

1- is studying French or Spanish

2- is studying Spanish or Italian

3- is studying one of the three languages

4- is not studying anyone of the three languages



SoL/

A={ the student Is studying French}

B={ the student is studying Spanish}

C={ the student is studying Italian}

$$\mathbf{1-}\ P(A\ or\ B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{45}{120} + \frac{30}{120} - \frac{14}{120} = \frac{61}{120}$$

$$\mathbf{2-}\ P(B\ or\ C) = P(B \cup C) = P(B) + P(C) - P(B \cap c)$$

$$= \frac{30}{120} + \frac{25}{120} - \frac{10}{120} = \frac{45}{120}$$

$$\mathbf{3-}\ P(A\ or\ B\ or\ C) = P(A \cup B \cup C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= \frac{45}{120} + \frac{30}{120} + \frac{25}{120} - \frac{14}{120} - \frac{8}{120} - \frac{10}{120} + \frac{6}{120} = \frac{74}{120}$$

$$\mathbf{4-}\ P(A \cup B \cup C)^c = 1 - P(A \cup B \cup C)$$

$$= 1 - \frac{74}{120} = \frac{46}{120}$$

Example : Let two cards be selected from an ordinary deck of cards(52). find the probability of the following:

1- each card have one color

2- the two cards from the same group

3- at least one of them diamond

SoL/

$$1 - \boldsymbol{P(A)} = \frac{\boldsymbol{C_1^{26} C_1^{26}}}{\boldsymbol{C_2^{52}}} = \frac{\boldsymbol{676}}{\boldsymbol{1325}} = \mathbf{0.51}$$

$$2 - \boldsymbol{P(B)} = \frac{\boldsymbol{C_2^{13} + C_2^{13} + C_2^{13} + C_2^{13}}}{\boldsymbol{C_2^{52}}} = \frac{\boldsymbol{312}}{\boldsymbol{1325}} = \mathbf{0.236}$$

$$3 - \boldsymbol{P(C)} = \frac{\boldsymbol{C_1^{13} C_1^{39} + C_2^{13} C_0^{39}}}{\boldsymbol{C_2^{52}}} = \frac{\boldsymbol{507+78}}{\boldsymbol{1325}} = \mathbf{0.442}$$

Example: let a card be selected from an ordinary deck of cards(52)

Let A = { the card is a spade}

B = {the card is face card}

Compute $\boldsymbol{P(A)}, \boldsymbol{P(B)}$ & $\boldsymbol{P(A \cap B)}$

SoL/

$$1 - \boldsymbol{P(A)} = \frac{\boldsymbol{C_1^{13}}}{\boldsymbol{C_1^{52}}} = \frac{\boldsymbol{13}}{\boldsymbol{52}} = \mathbf{0.25}$$

$$2 - \boldsymbol{P(B)} = \frac{\boldsymbol{C_1^{12}}}{\boldsymbol{C_1^{52}}} = \frac{\boldsymbol{12}}{\boldsymbol{52}} = \mathbf{0.23}$$

$$\mathbf{3} - \boldsymbol{P(A \cap B)} = \frac{\boldsymbol{n(A \cap B)}}{\boldsymbol{n}} = \frac{\boldsymbol{C_1^3}}{\boldsymbol{C_1^{52}}} = \frac{\boldsymbol{3}}{\boldsymbol{52}} = \mathbf{0.058}$$

$\boldsymbol{P(A \cup B)}$? $\boldsymbol{H.W.}$

Example: Let $\boldsymbol{P}$ be a probability function on $\boldsymbol{S} = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$, Find $\boldsymbol{P}(\boldsymbol{a_1})$ If :

1_ $\boldsymbol{P}(\boldsymbol{a_2}) = \frac{\mathbf{1}}{\mathbf{3}}$ & $\boldsymbol{P}(\boldsymbol{a_3}) = \frac{\mathbf{1}}{\mathbf{4}}$

2_ $\boldsymbol{P}(\boldsymbol{a_2}) = \mathbf{3}\boldsymbol{P}(\boldsymbol{a_1})$ & $\boldsymbol{P}(\boldsymbol{a_3}) = \frac{\mathbf{1}}{\mathbf{5}}$

3_ $\boldsymbol{P}(\{\boldsymbol{a_2}, \boldsymbol{a_3}\}) = \mathbf{2}\boldsymbol{P}(\boldsymbol{a_1})$

4_ $\boldsymbol{P}(\boldsymbol{a_3}) = \mathbf{2}\boldsymbol{P}(\boldsymbol{a_2})$ & $\boldsymbol{P}(\boldsymbol{a_2}) = \mathbf{3}\boldsymbol{P}(\boldsymbol{a_1})$

Sol/

1_ $\boldsymbol{P}(\boldsymbol{a_1}) + \boldsymbol{P}(\boldsymbol{a_2}) + \boldsymbol{P}(\boldsymbol{a_3}) = \mathbf{1}$

$\rightarrow \boldsymbol{P}(\boldsymbol{a_1}) + \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{1}}{\mathbf{4}} = \mathbf{1}$

$\rightarrow \boldsymbol{P}(\boldsymbol{a_1}) = \frac{\mathbf{5}}{\mathbf{12}}$

2_ $\boldsymbol{P}(\boldsymbol{a_1}) + \boldsymbol{P}(\boldsymbol{a_2}) + \boldsymbol{P}(\boldsymbol{a_3}) = \mathbf{1}$

$\boldsymbol{P}(\boldsymbol{a_1}) + \mathbf{3}\boldsymbol{P}(\boldsymbol{a_1}) + \frac{\mathbf{1}}{\mathbf{5}} = \mathbf{1}$

$\rightarrow 4\boldsymbol{P}(\boldsymbol{a_1}) = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{5}} = \frac{\mathbf{4}}{\mathbf{5}}$

$\therefore \boldsymbol{P}(\boldsymbol{a_1}) = \frac{\mathbf{1}}{\mathbf{5}}$

3_ $\boldsymbol{P}(\boldsymbol{a_1}) + \boldsymbol{P}(\boldsymbol{a_2}) + \boldsymbol{P}(\boldsymbol{a_3}) = \mathbf{1}$

$\boldsymbol{P}(\boldsymbol{a_1}) + \mathbf{2}\boldsymbol{P}(\boldsymbol{a_1}) + \mathbf{2}\boldsymbol{P}(\boldsymbol{a_1}) = \mathbf{1}$

$\rightarrow \mathbf{5}\,\boldsymbol{P}(\boldsymbol{a_1}) = \mathbf{1}$

$\rightarrow \boldsymbol{P}(\boldsymbol{a_1}) = \frac{\mathbf{1}}{\mathbf{5}} = \mathbf{0.2}$

4_ $\boldsymbol{P}(\boldsymbol{a_1}) + \boldsymbol{P}(\boldsymbol{a_2}) + \boldsymbol{P}(\boldsymbol{a_3}) = \mathbf{1}$

$\boldsymbol{P}(\boldsymbol{a_1}) + \mathbf{3}\boldsymbol{P}(\boldsymbol{a_1}) + \mathbf{2}[\mathbf{3}\boldsymbol{P}(\boldsymbol{a_1})] = \mathbf{1}$

$\rightarrow \mathbf{10}\,\boldsymbol{P}(\boldsymbol{a_1}) = \mathbf{1}$

$\rightarrow \boldsymbol{P}(\boldsymbol{a_1}) = \frac{\mathbf{1}}{\mathbf{10}} = \mathbf{0.1}$

Example: A box contains 4 white , 6 red , 5 blue and 3 orange balls. Three balls are drawn at random from the box ; what is the probability that :

a_ Two of them are white ,

b_ one is red and two are orange ,

c_ none of them are blue ,

d_ all of them are either white or blue .

Sol/

$$1 - \boldsymbol{P(A) = \frac{C_2^4 C_1^{14}}{C_3^{18}} = \frac{6*14}{816} = \frac{84}{816} = 0.103}$$

$$2 - \boldsymbol{P(B) = \frac{C_1^6 C_2^3}{C_3^{18}} = 0.0221}$$

$$3 - \boldsymbol{P(C) = \frac{C_3^{13}}{C_3^{18}} = 0.351}$$

$$4 - \boldsymbol{P(B) = \frac{C_3^4 + C_3^5}{C_3^{18}} = 0.0172}$$

Example: Two cards are drawn one after the other without replacement from a well-shuffled deck of 52 cards ; Find the probability that they are :

A_ of spades

B_ one is heart and the other is diamond

C_ number 10 of spade and Jack of club

D_ red cards

E_ numbered cards

F_ picture cards

Sol/

ملاحظه: عندما يذكر بالسؤال without replacement بدون ارجاع أي عملية السحب والاختيار تكون بدون ارجاع فإن العدد يتناقص، وإذا لم يذكر في السؤال (السحب او الاختيار بدون ارجاع) فإن العدد يبقى كما هو او يحل شيء محل اخر. وسيتم توضيح ذلك في حل السؤال التالي:

Without replacement	with replacement
$P(A) = \frac{C_1^{13}}{C_1^{52}} * \frac{C_1^{12}}{C_1^{51}} = \frac{13}{52} * \frac{12}{51} =$	$P(A) = \frac{C_2^{13}}{C_2^{52}}$
$P(B) = \frac{C_1^{13}}{C_1^{52}} * \frac{C_1^{13}}{C_1^{51}} = \frac{13}{52} * \frac{13}{51} =$	$P(B) = \frac{C_1^{13} * C_1^{13}}{C_2^{52}}$
$P(C) = \frac{C_1^{1}}{C_1^{52}} * \frac{C_1^{1}}{C_1^{51}} = \frac{1}{52} * \frac{1}{51} =$	$P(C) = \frac{C_1^{1} * C_1^{1}}{C_2^{52}}$
$P(D) = \frac{C_1^{26}}{C_1^{52}} * \frac{C_1^{25}}{C_1^{51}} = \frac{26}{52} * \frac{25}{51} =$	$P(D) = \frac{C_2^{26}}{C_2^{52}}$
$P(E) = \frac{C_1^{40}}{C_1^{52}} * \frac{C_1^{39}}{C_1^{51}} = \frac{40}{52} * \frac{39}{51} =$	$P(E) = \frac{C_2^{40}}{C_2^{52}}$
$P(F) = \frac{C_1^{12}}{C_1^{52}} * \frac{C_1^{11}}{C_1^{51}} = \frac{12}{52} * \frac{11}{51} =$	$P(F) = \frac{C_2^{12}}{C_2^{52}}$

Example: A card is selected at random from an ordinary deck of 52 cards , let

A= {the card is spade}

B= {the card is a picture card; i.e., a Jack, Queen and king}

Find $P(A)$, $P(B)$ & $P(A \cup B)$

Sol/

$$P(A) = \frac{C_1^{13}}{C_1^{52}} = \frac{13}{52} \quad ; \quad P(B) = \frac{C_1^{12}}{C_1^{51}} = \frac{12}{51}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore P(A \cap B) = \{ the\ card\ is\ picture\ card\ of\ spade \}$$

$$\therefore P(A \cap B) = \frac{C_1^{3}}{C_1^{52}} = \frac{3}{52}$$

$$\therefore P(A \cup B) = \frac{13}{52} + \frac{12}{51} - \frac{3}{52} = \frac{22}{52}$$

Example: Two prizes are to be given to two students who are chosen at random from the students [Ayad, Bashir, Khalid, Mazin and Sami];

1- Find elements of the sample space

2- Find the probability of the events

E_1 : Ayad receives a prize

E_2 : either Bashir or Sami receives a prize

E_3 : Khalid dose not receives a prize

E_4 : either Khalid or Mazin receives a prize but not both

Sol/

$$\boldsymbol{C_2^5 = \frac{5!}{2! * 3!} = 10} \quad ways$$

1- $\therefore \boldsymbol{S = \{AB, AK, AM, AS, BK, BM, BS, KM, KS, MS\}}$

2- $E_1 = \{\boldsymbol{AB, AK, AM, AS}\}$

$$\therefore P(E_1) = \frac{\boldsymbol{n(E_1)}}{\boldsymbol{n}} = \boldsymbol{\frac{4}{10}} \quad or \quad \boldsymbol{\frac{C_1^4}{C_2^5}}$$

$E_2 = \{\boldsymbol{B, S}\}$ &

$\boldsymbol{B = \{AB, BK, BM, BS\}} \quad S = \{\boldsymbol{AS, BS, KS, MS}\}$

$$\therefore P(B) = \frac{\boldsymbol{n(B)}}{\boldsymbol{n}} = \boldsymbol{\frac{4}{10}} \quad ; \quad \therefore P(S) = \frac{\boldsymbol{n(S)}}{\boldsymbol{n}} = \boldsymbol{\frac{4}{10}}$$

$$\therefore (B \cap S) = \{BS\} \quad \rightarrow P(B \cap S) = \boldsymbol{\frac{1}{10}}$$

$$\therefore P(B \cup S) = P(B) + P(S) - P(B \cap S) = \boldsymbol{\frac{4}{10} + \frac{4}{10} - \frac{1}{10} = \frac{7}{10}}$$

$E_3 = \{AB, AM, AS, BM, BS, MS\}$

$$\therefore P(E_3) = \frac{\boldsymbol{n(E_3)}}{\boldsymbol{n}} = \boldsymbol{\frac{6}{10}}$$

$E_4 = \{K, M\}$ & $K = \{AK, BK, KS\}$ $M = \{AM, BM, MS\}$

$$\therefore P(K \cup M) = P(K) + P(M) = \boldsymbol{\frac{3}{10} + \frac{3}{10} = \frac{6}{10}}$$

4-3 : Independent Events الحوادث المستقلة

Two events A & B are independent iff :-

$$P(A \cap B) = P(A) * P(B)$$

Then :

Any two events A & B the probability :

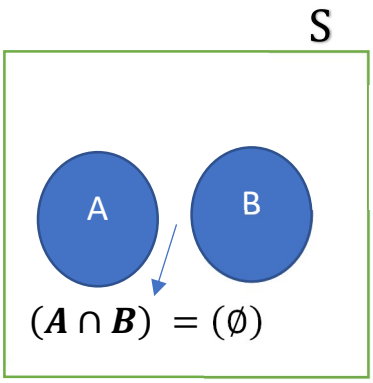
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If : - هما الحدثان اللذان لا يمكن ان يحدثا في آن واحد

Mutually exclusive : $P(A \cup B) = P(A) + P(B) - 0$

حوادث متنافية

$$\therefore P(A \cap B) = P(\emptyset) = 0$$



If : Independent events

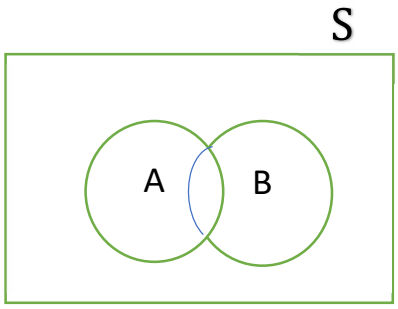
يقال لحدثان انهما مستقلان اذا كان احتمال ظهور احدهما لا يتأثر بحدوث الحدث الاخر : -

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\because P(A \cap B) = P(A) * P(B)$$

في حال تحقق الشرط أعلاه يقال ان الحدثان مستقلان وان

احتمال الاتحاد كما يلي :-



$$P(A \cup B) = P(A) + P(B) - P(A) * P(B)$$

Example: Suppose that two machines 1 & 2 are operated in a factory independently of each other . Let $\boldsymbol{A}$ be the event that machine 1 will become inoperative during a given 8- hour period , let $\boldsymbol{B}$ be the event that machine 2 will become inoperative during the same period, and suppose that $\boldsymbol{P(A) = \frac{1}{3}}$ and $\boldsymbol{P(B) = \frac{1}{4}}$. we shall determine the probability that at least one of the machines will become inoperative during a given period .

The probability $\boldsymbol{P(A \cap B)}$; that both machines will become inoperative during the period is :

$$\boldsymbol{P(A \cap B) = P(A) * P(B) = \frac{1}{3} * \frac{1}{4} = \frac{1}{12}}$$

Therefore, the probability $\boldsymbol{P(A \cup B)}$ that at least one of machines will become inoperative during the period is :

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$\boldsymbol{= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{6}{12} = \frac{1}{2}}$$

-.-

A & B are independent

$$\boldsymbol{P(A \cap B) = P(A) * P(B)}$$

$$\boldsymbol{P(A) * P(B) = P(A \cap B)}$$

A & B are not independent

$$\boldsymbol{P(A \cap B) \neq P(A) * P(B)}$$

Example: $\boldsymbol{S = \{1, 2, 3, 4, 5, 6\}}$

$\boldsymbol{A = \{2, 4, 6\}}$ & $\boldsymbol{B = \{1, 2, 3, 4\}}$

Are $\boldsymbol{A}$ & $\boldsymbol{B}$ independent ?

Sol/

$$\boldsymbol{P(A) = \frac{3}{6}} \quad ; \quad \boldsymbol{P(B) = \frac{4}{6}}$$

$$\boldsymbol{(A \cap B) = \{2, 4\}} \quad ; \quad \boldsymbol{P(A \cap B) = \frac{2}{6} = \frac{1}{3}}$$

$$\boldsymbol{P(A \cap B) = P(A) * P(B)}$$

$$\boldsymbol{\frac{1}{3} = \frac{3}{6} * \frac{4}{6} \rightarrow \frac{1}{3} = \frac{1}{3}}$$

$\therefore$ A & B are independent events .

- **Theorem :-**

If A & B are two independent events then :-

1- A^c & B are also independent.

2- A & B^c are also independent.

3- A^c & B^c are also independent.

Proof/

1- $\boldsymbol{P(A^c \cap B) = P(A^c) * P(B)}$

$\because$ $\boldsymbol{P(A^c \cap B) = P(B/A) = P(B) - P(A \cap B)}$

$$\boldsymbol{= P(B) - [P(A) * P(B)]}$$

$$\boldsymbol{= P(B)[1 - P(A)]}$$

$$\boldsymbol{= P(B) * P(A^c)}$$

2- $\boldsymbol{P(A \cap B^c) = P(A) * P(B^c)}$

$\because$ $\boldsymbol{P(A \cap B^c) = P(A/B) = P(A) - P(A \cap B)}$

$$\boldsymbol{= P(A) - [P(A) * P(B)]}$$

$$\boldsymbol{= P(A)[1 - P(B)]}$$

$$\boldsymbol{= P(A) * P(B^c)}$$

3- $\boldsymbol{P(A^c \cap B^c) = P(A^c) * P(B^c)}$

$\boldsymbol{P(A^c \cap B^c) = P(A \cup B)^c}$ حسب قانون دي موركان

$$= \mathbf{1} - \boldsymbol{P(A \cup B)}$$

$$= \mathbf{1} - [\boldsymbol{P(A) + P(B) - P(A \cap B)}]$$

$$= \mathbf{1} - [\boldsymbol{P(A) + P(B) - P(A) * P(B)}]$$

$$= [\mathbf{1} - \boldsymbol{P(A)}] - \boldsymbol{P(B) + P(A) * P(B)}$$

$$= \boldsymbol{P(A^c) - P(B)}[\mathbf{1} - \boldsymbol{P(A)}]$$

$$= \boldsymbol{P(A^c) - P(B)P(A^c)}$$

$$= \boldsymbol{P(A^c)}[\mathbf{1} - \boldsymbol{P(B)}]$$

$\therefore \boldsymbol{P(A^c \cap B^c) = P(A^c) * P(B^c)}$

-.-

If A , B & C are three events , we said that A , B & C are independent events iff: -

1- $\boldsymbol{P(A \cap B) = P(A) * P(B)}$

1- $\boldsymbol{P(A \cap C) = P(A) * P(C)}$

1- $\boldsymbol{P(B \cap C) = P(B) * P(C)}$

1- $\boldsymbol{P(A \cap B \cap C) = P(A) * P(B) * P(C)}$

If only three satisfy then we said A , B & C are pairwise-independent.

بمعنى مستقلة مثنى - مثنى (زوج - زوج)

Example: Let $\boldsymbol{S} = \{\mathbf{123}, \mathbf{132}, \mathbf{321}, \mathbf{312}, \mathbf{231}, \mathbf{213}, \mathbf{111}, \mathbf{222}, \mathbf{333}\}$ and each of the nine elementary events in $\boldsymbol{S}$ occurs with probability $1/9$. Let $\boldsymbol{A_k}$ be the event that the k-th digit is 1 , $k = 1,2,3$ then :-

Sol/

$\boldsymbol{A_1} = \{\mathbf{111}, \mathbf{123}, \mathbf{132}\}$

$\rightarrow \boldsymbol{P(A_1)} = \frac{\mathbf{3}}{\mathbf{9}} = \frac{\mathbf{1}}{\mathbf{3}}$

$$A_2 = \{111, 213, 312\}$$

$$\rightarrow P(A_2) = \frac{3}{9} = \frac{1}{3}$$

$$A_3 = \{111, 321, 231\}$$

$$\rightarrow P(A_3) = \frac{3}{9} = \frac{1}{3}$$

$$A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \{111\}$$

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = 1/9$$

Then :

$$P(A_1 \cap A_2) = P(A_1) * P(A_2) = \frac{1}{3} * \frac{1}{3} = \frac{1}{9}$$

And

$$P(A_1 \cap A_3) = P(A_1) * P(A_3) = \frac{1}{3} * \frac{1}{3} = \frac{1}{9}$$

And

$$P(A_2 \cap A_3) = P(A_2) * P(A_3) = \frac{1}{3} * \frac{1}{3} = \frac{1}{9}$$

$\therefore$ A , B & C are pairwise-independent.

$$A_1 \cap A_2 \cap A_3 = \{111\}$$

$$\therefore P(A_1 \cap A_2 \cap A_3) = \frac{n(A_1 \cap A_2 \cap A_3)}{n} = 1/9$$

And

$$\rightarrow P(A_1 \cap A_2 \cap A_3) = P(A_1) * P(A_2) * P(A_3)$$

$$\rightarrow \quad \frac{1}{9} \quad = \frac{1}{3} * \frac{1}{3} * \frac{1}{3}$$

$$\rightarrow \quad \frac{1}{9} \quad \neq \quad \frac{1}{27}$$

$\therefore$ A , B & C are not independent.