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Karmarkars Method

Karmarkars proposed a new method in (1984) for solving large–scale linear programming problems very efficiently. The method is known as an interior method since it finds improved search directions strictly in the interior of the feasible space. In fact, it was found that Karmarkars method is as much as (50) times faster than the simplex method for large problems.

Statement of the problems:

Karmarkars method requires the L.P.P in the following form:

Min $f=CX$

S.t

$AX=0$

$x_1+x_2+\cdots x_n=1$

$x_1,x_2,\cdots x_n\geq 0$

where $X=\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, $C=\begin{bmatrix} c_1 & c & \cdots & c_n \end{bmatrix}$

A is an **m*n** matrix

Algorithm:

1. Being at the feasible point **$x_1=\begin{bmatrix} 1/n & 1/n & \cdots & 1/n \end{bmatrix}^T$** set **$k=1$** .
2. Test for optimality **(if $(CX)_k<\varepsilon$)** where **(ε)** is small number step otherwise go to step 3.
3. Compute the new point **$X_{(k+1)}$** . For this, we first find a point **$y_{(k+1)}$** in the transformed unit simples as:

$$y_{(k+1)}=\begin{bmatrix} 1/n & 1/n & \cdots & 1/n \end{bmatrix}^T-\frac{\propto C_p}{\|C_p\|\sqrt{n(n-1)}}$$

where:

$\|C_p\|$ Is the length of the vector **C_p.**

(I) The identity matrix of order **n.**

$[DX_k]$ on $(n*n)$ matrix with all off-diagonal entries equal to (0) and diagonal entries equal to the components of the vector (X_k).

$P=\begin{bmatrix} A(DX_k) \\ 1 \end{bmatrix}$

And the value of the parameters $(\propto)$ is usually chosen as $(\propto=\frac{1}{4})$ to ensure convergence. Then the new point $(X_{(k+1)})$ are determined as: -

$X_{(K+1)}=\frac{DX_k \quad y_{(k+1)}}{1 \; DX_k \quad y_{(k+1)}}$

Set the new iteration number as $(K=K+1)$ and go to step 2.

Example: find the solution of the following problem using Karmakers method.

$\min f=2x_1+x_2-x_3$

S.t

$x_2-x_3=0$

$x_1+x_2+x_3=1$

$x_1,x_2,x_3\geq 0$

Use the value $(\in=0.05)$ for testing the convergence of the procedure.

sol:

$x_1=\begin{bmatrix} 1/n \\ 1/n \\ 1/n \end{bmatrix}=\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

هذه المتجهة تُحقق القيد الأول:

$x_2-x_3=0$

$1/3-1/3=0$

$CX_k=f(x_1)=2*1/2+1/3+1/3=2/3>\in$

$A=\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \quad ,C=\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$

Iteration: (k=1)

$Dx_1=\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

$A(Dx_1)=\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}=\begin{bmatrix} 0 & 1/3 & \frac{-1}{3} \end{bmatrix}$

$C(Dx_1)=\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}=\begin{bmatrix} \frac{2}{3} & 1/3 & \frac{-1}{3} \end{bmatrix}$

$P\begin{bmatrix} A(Dx_1) \\ 1 \end{bmatrix}=\begin{bmatrix} 0 & 1/3 & \frac{-1}{3} \\ 1 & 1 & 1 \end{bmatrix}$

$P\,P^T=\begin{bmatrix} 0 & 1/3 & \frac{-1}{3} \\ 1 & 1 & 1 \end{bmatrix}*\begin{bmatrix} 0 & 1 \\ 1/3 & 1 \\ \frac{-1}{3} & 1 \end{bmatrix}=\begin{bmatrix} 2/9 & 0 \\ 0 & 3 \end{bmatrix}$

$(P\,P^2)^{-1}=\begin{bmatrix} 2/9 & 0 \\ 0 & 1/3 \end{bmatrix}$

$P'(P\,P')^{-1}P=\begin{bmatrix} 0 & 1 \\ 1/3 & 1 \\ \frac{-1}{3} & 1 \end{bmatrix}*\begin{bmatrix} 2/9 & 0 \\ 0 & 1/3 \end{bmatrix}*\begin{bmatrix} 0 & 1/3 & \frac{-1}{3} \\ 1 & 1 & 1 \end{bmatrix}$

=[■(0&1/3@3/2&1/3@(-3)/2&1/3)]*[■(0&1/3&(-1)/3 @1&1&1)] □(→┬( )) [■(1/3&1/3&1/3@1/3&5/6&(-1)/6@1/3&(-1)/6&5/6)]

C_p=[I-P^T (PP^T)^(-1) P] [C(Dx_1)]

=[[■(1&0&0@0&1&0@0&0&1)]-[■(1/3&1/3&1/3@1/3&5/6&(-1)/6 @1/3&(-1)/6&5/6)]]*[■(2/3@1/3@(-1)/3)] →┬ =[■(2/3&(-1)/3&(-1)/3@ (-1)/3&1/6&1/6@(-1)/3&1/6&1/6)]*[■(2/3@1/3@(-1)/3)]=[■(4/9@(-2)/9@(-2)/9)]

|(|C_p |)|=√((4/9)^2+((-2)/9)^2+((-2)/9)^2)=√(24/81)

y_2=[■(1/3@1/3@1/3)]-(∝C_p)/(|(|C_p |)| √(n(n-1)))

=[■(1/3@1/3@1/3)]-(1)/(4) [■(4/9@(-2)/9@(-2)/9)] 1/(√(24/81) √6)

=[■(1/3@1/3@1/3)]-[■(1/9@(-1)/18@(-1)/18)] 1/√(144/81) →┬( )=[■(1/3@1/3@1/3)]-[■(1/9@ (-1)/18 @(-1)/18)] (9/12)

y_2=[■(1/3@1/3@1/3)] - [■(1/12@(-1)/24@(-1)/24)]=[■(1/4@(3)/8@3/8)]

(Dx_1) y_2=[■(1/3&0&0@0&1/3&0@0&0&1/3)]*[■(1/4@3/8@3/8)]=[■(1/12@1/8@1/8)]
1(Dx_1) y_2=[■(1&1&1)]*[■(1/12@1/8@1/8)]=1/12+1/8+1/8=1/3

x_2=((Dx_1) y_2)/(1 (Dx_1) y_2)=[■(1/12@1/8@1/8)]/(1/3)=[■(1/4@3/8@3/8)]
f(x_2)=2(1/4)+3/8-3/8=1/2<f(x_1)>ϵ

Iteration 2: -
Dx_2=[■(1/4&0&0@0&3/8&0@0&0&3/3)]
A(Dx_2)=[■(0&1&-1)]*[■(1/4&0&0@0&3/8&0@0&0&3/3)]=[■(0&3/8&(-3)/8)]
C(Dx_2)=[■(2&1&-1)]*[■(1/4&0&0@0&3/8&0@0&0&3/3)]=[■(1/2&3/8&(-3)/8)]
P=[■(A(Dx_2)@1)]=[■(0&3/8&(-3)/8@1&1&1)]

P P^T=[■(0&3/8&(-3)/8@1&1&1)]*[■(0&1@3/8&1@(-3)/8&1)] =[■(9/32&0@0&3)]

(P P^T)^(-1)=[■(32/9&0@0&1/3)]

P′(P P′)^(-1) P=[■(0&1@3/8&1@(-3)/8&1)]*[■(32/9&0@0&1/3)]*[■(0&3/8&(-3)/8@1&1&1)]

=[■(0&1/3@4/3&1/3@(-4)/3&1/3)]*[■(0&3/8&(-3)/8@1&1&1)]= [■(1/3&1/3&1/3@1/3&5/6&(-1)/6@1/3&(-1)/6&5/6)]

C_p=[I-P^T (PP^T)^(-1) P] [C(Dx_2)]^T

=[[■(1&0&0@0&1&0 @0&0&1)]- [■(1/3&1/3&1/3@1/3&5/6&(-1)/6@1/3&(-1)/6&5/6)]]* [■(1/2@3/8@(-3)/8)]=[■(2/3&(-1)/3&(-1)/3@(-1)/3&1/6&1/6@(-1)/3&1/6&1/6)]*[■(1/2@3/8@(-3)/8)]=[■(1/3@(-1)/6@(-1)/6)]

|(|C_p |)|=√((1/3)^2+((-1)/6)^2+((-1)/6)^2) =√(6/36)

y_3=[■(1/n@1/n@1/n)]-(∝ C_p)/(|(|C_p |)| √(n(n-1)))
=[■(1/3@1/3@1/3)]-(1/4 [■(1/3@(-1)/6@(-1)/6)])/(√(6/36) √6) →┬( )
[■(1/3@1/3@1/3)]-[■(1/12@(-1)/24@(-1)/24)] = [■(1/4@3/8@3/8)]

(Dx_2) y_3=[■(1/4&0&0@0&3/8&0@0&0&3/8)]*[■(1/4@3/8@3/8)]= [■(1/16@9/64@9/64)]

1(Dx_2) y_3=[■(1&1&1)]*[■(1/16@9/64@9/64)]=11/32
x_3=((Dx_2) y_3)/(1 (Dx_2) y_3)

=[■(1/16@9/64@9/64)]*32/11 →┬( ) [■(2/11@9/22@9/22)]

f(x_3)=2(2/11)+9/22-9/22 →┬( )=4/11<f(x_2)>ϵ

Example: find the solution of the following problem suing karmarkers method?

min Z=x_1+3x_2±3x_3

S.t

x_2-x_3=0

x_1+x_2+x_3=1

x_1 , x_2 , x_3≥0

sol:
x_1=[■(1/n@1/n@1/n)]=[■(1/3@1/3@1/3)]

x_2-x_3=0

1/3-1/3=0

و هذا المتجه يحقق القيد الأول

CX_1=f(x_1)=1/3+3/3-3/3=|1/3|

A=[■(0&1&-1)] , C=[■(1&3&-3)]

iteration (1):

Dx_1=[■(1/3&0&0@ 0&1/3&0 @0&0&1/3)]

A(Dx_1)=[■(0&1&-1)]*[■(1/3&0&0@ 0&1/3&0 @0&0&1/3)]→[■(0&1/3&-1/3)]

C(Dx_1)=[■(1&3&-3)]*[■(1/3&0&0@ 0&1/3&0 @0&0&1/3)]→[■(1/3&1&-1)]

P=[■(A(Dx_1)@1)]=[■(0&1/3&(-1)/3@1&1&1)]

P P^T=[■(0&1/3&(-1)/3@1&1&1)]*[■(0&1@1/3&1@(-1)/3&1)] □(→┬( )) =[■(9/2&0@0&3)]

(P P^T)^(-1)=[■(9/2&0@0&1/3)]

P^T (P P^T)^(-1) P=[■(0&1@1/3&1@(-1)/3&1)]*[■(9/2&0@0&1/3)]*[■(0&1/3&(-1)/3@1&1&1)]

=[■(0&1/3@3/2&1/3@(-3)/2&1/3)]*[■(0&1/3&(-1)/3@1&1&1)] →┬( )=[■(1/3&1/3&1/3@1/3&5/6&(-1)/6@1/3&(-1)/6&5/6)]

C_p=[I-P^T (PP^T)^(-1) P] [C(Dx_1)]^T

= [[■(1&0&0@0&1&0@0&0&1)]-[■(1/3&1/3&1/3@1/3&5/6&(-1)/6@1/3&(-1)/6&5/6)]]*[■(1/3@1@-1)]

=[■(2/3&(-1)/3&(-1)/3@(-1)/3&1/6&1/6@(-1)/3&1/6&6/6)]*[■(1/3@1@-1)]=[■(2/9@(-1)/9@(-1)/9)]

|(|C_p |)|=√((2/9)^2+((-1)/9)^2+((-1)/9)^2) □(→┬( )) √(4/81+1/81+1/81)=√(6/81)

y_2=[■(1/n@1/n@1/n)]-(∝ C_p)/(|(|C_p |)| √(n(n-1)))

=[■(1/3@1/3@1/3)]-1/4*[■(2/9@(-1)/9@(-1)/9)] →=1/(√(6/81) √6)

=[■(1/3@1/3@1/3)]-[■(1/18@(-1)/36@(-1)/36)] 3/2 □(→┬( )) =[■(1/3@1/3@1/3)]-[■(1/18@(-1)/24@(-1)/24)]=**[■(1/4@3/8@3/3)]**

(Dx_1) y_2=[■(1/3&0&0@ 0&1/3&0 @0&0&1/3)]*[■(1/4@3/8@3/3)]=[■(1/12@1/8@1/8)]

1(Dx_1) y_2=[■(1&1&1)]*[■(1/12@1/8@1/8)] =1/3

x_2=((Dx_1) y_2)/(1(Dx_1) y_2)=[■(1/4@1/8@3/8)] f(x_2)=1/4+3 3/8-3 3/8 =1/4

iteration (2):

Dx_2=[■(1/4&0&0@ 0&3/8&0@0&0&3/8)]

A(Dx_2)=[■(0&1&-1)]*[■(1/4&0&0@ 0&3/8&0@0&0&3/8)]=→[■(0&3/8&(-3)/8)]

C(Dx_2)=[■(1&3&-3)]*[■(1/4&0&0@ 0&3/8&0@0&0&3/8)]=[■(1/4&9/8&(-9)/8)]

P=[■(A Dx_2@1)]=[■(0&3/8&(-3)/8@1&1&1)]

(P P^T)=[■(9/32&0@0&3)] □(→┬(   )) (P P^T)^(-1)=[■(32/9&0@0&1/3)]

P^T (P P^T)^(-1) P=[■(1/3&1/3&1/3@1/3&5/6&(-1)/6@1/3&(-1)/6&5/6)]

C_p=[I-P^T (P P^T)^(-1) P] [C(Dx_1)]^T

=[■(2/3&(-1)/3&(-1)/3@(-1)/3&1/6&1/6@(-1)/3&1/6&1)]*[■(1/4@9/8@(-9)/8)]=[■(1/6@(-1)/12@(-1)/12)]

|(|C_p |)|=√((1/6)^2+((-1)/12)^2+((-1)/12)^2   )   □(→┬(   )   ) √(1/36+1/144+1/144)=√(6/144)

y_3=[■(1/n@1/n@1/n)]-(∝ C_p)/(|(|C_p |)| √(n(n-1)))   →┬(   )=[■(1/3@1/3@1/3)]-1/4*[■(1/6@(-1)/12@(-1)/12)] 1/(√(6/144) √6)

=[■(1/3@1/3@1/3)]-[■(1/24@(-1)/48@(-1)/48)]   12/6   →┬(   ) [■(1/3@1/3@1/3)]-[■(1/24@(-1)/48@(-1)/48)]=[■(1/4@3/8@3/8)]

(Dx_2   )   y_3=[■(1/4&0&0@   0&3/8&0@0&0&3/8)]*[■(1/4@3/8@3/8)   ]=[■(1/16@9/64@9/64)]

1(Dx_2) y_3=11/32

x_3=((Dx_2) y_3)/(1 (Dx_2) y_3)=[■(2/11@9/22@9/22)]

f(x_3)=2/11+3 9/22-3 9/22=2/11<f(x_2).

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