Lagrange method
طريقة لاكرانج

Theorem: Let f and g be functions of with continuous first partial derivatives on some open set containing the constraint (قيد) curve $g(x,y)=0$ and assume that $\nabla g \neq 0$ at any point on this curve. If f has a constraint relative extrema, then this extrema occur (يحدث) at a point(x_o,y_o) on the constraint curve at which the gradient vectors $\nabla f(x_o,y_o)$ and

$$\nabla f(x_o,y_o) = \lambda \ \nabla g(x_o,y_o)$$

Example 1: Find the extrema point of the function $f = z^2 - x^2 - y^2 - 1$ subject to the constraint $2x + 2y - 3z + 1 = 0$, use Lagrange method.

Solution: Let $f = z^2 - x^2 - y^2 - 1$ and $g = 2x + 2y - 3z + 1 = 0$

$$\nabla f = \lambda \ \nabla g(x,y,z)$$

$$\nabla f(x,y,z) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k$$
$$= -2xi - 2yj + 2zk$$

$$\nabla g(x,y,z) = g_x(x,y,z)i + g_y(x,y,z)j + g_z(x,y,z)$$
$$= 2i + 2yj - 3k$$

$$\therefore \quad -2xi - 2yj + 2zk = \lambda \ (2i + 2j - 3k)$$

$$-2x = 2\lambda \ \ldots\ldots\ldots(1)$$
$$-2y = 2\lambda \ \ldots\ldots\ldots(2)$$
$$2z = -3\lambda \ \ldots\ldots\ldots(3)$$

From Eq.(1) we get $x = -\lambda$ ………(4)
From Eq.(2) we get $y = -\lambda$ ………(5)
From Eq.(3) we get $z = -\frac{3}{2}\lambda$ ……..(6)
Substitute (4),(5),(6) in constraint $2x + 2y - 3z + 1 = 0$

$$2(-\lambda) + 2(-\lambda) - 3\left(-\frac{3}{2}\lambda\right) + 1 = 0$$

$$-2\lambda - 2\lambda + \frac{9}{2}\lambda = -1 \quad \Rightarrow \quad -4\lambda + \frac{9}{2}\lambda = -1 \Rightarrow \quad \frac{1}{2}\lambda = -1$$

$$\therefore \quad \lambda = -2 \ \ldots\ldots\ldots\ldots(*).$$

Substitute above this value in Eq. (4),(5),(6) we obtain the following :
$x = 2$, $y = 2$, $z = 3$
The extrema point is p (2,2,3).

Example 2: Find the extrema points of the function $f = xy$, subject to the constraint $x^2 + y^2 - 10 = 0$, use Lagrange method.

Solution: Let $f = xy$ and $g = x^2 + y^2 - 10 = 0$

$$\nabla f(x,y) = \lambda\ \nabla g(x,y)$$
$$yi + xj = \lambda\,(2xi + 2yj)$$

$$y = 2x\lambda \ \Rrightarrow\ \lambda = \frac{y}{2x} \ \ldots\ldots\ldots(1)$$
$$x = 2y\lambda \ \Rrightarrow\ \lambda = \frac{x}{2y} \ \ldots\ldots\ldots(2)$$

From Eq. (1) and Eq. (2) we get $\frac{y}{2x} = \frac{x}{2y} \Rrightarrow \ 2x^2 = 2y^2$

$$\Rrightarrow \ x^2 = y^2 \ \ldots\ldots\ldots(3)$$

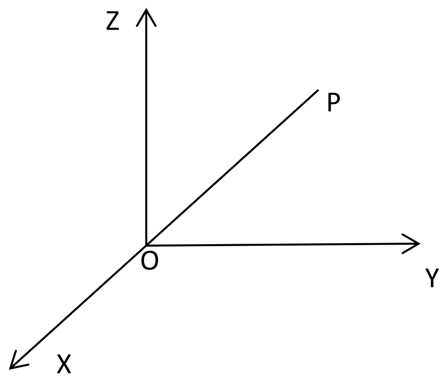
Substitute (3) in constraint $x^2 + y^2 - 10 = 0$

$$x^2 + x^2 - 10 = 0 \quad \Rrightarrow\ 2x^2 = 10 \quad \Rrightarrow x = \pm\sqrt{5}$$

Therefore $y = \pm\sqrt{5}$ and the extrema points are $p_1(-\sqrt{5}, -\sqrt{5}), p_2(\sqrt{5}, \sqrt{5})$.

Example 3: Find the nearest point in the plane $2x + 3y - z = 6$ of the origin, using Lagrange method

Solution:



$$\|OP\| = \sqrt{x^2 + y^2 + z^2}$$
$$f = x^2 + y^2 + z^2$$
$$g = 2x + 3y - z - 6 = 0$$
$$\nabla f(x,y,z) = \lambda\ \nabla g(x,y,z)$$
$$2xi + 2yj + 2zk = \lambda\,(2i + 3j - k)$$

$$2x = 2\lambda \ \Rrightarrow\ x = \lambda \ \ldots\ldots\ldots(1)$$
$$2y = 3\lambda \ \Rrightarrow\ y = \frac{3}{2}\lambda \ \ldots\ldots\ldots(2)$$
$$2z = -\lambda \ \Rrightarrow\ z = -\frac{1}{2}\lambda \ \ldots\ldots(3)$$

Substitute (1),(2),(3) in constraint $g = 2x + 3y - z - 6 = 0$

$$\Rrightarrow 2(\lambda) + 3\left(\frac{3}{2}\lambda\right) - \left(-\frac{1}{2}\lambda\right) - 6 = 0 \quad \Rrightarrow \ 2\lambda + \frac{9}{2}\lambda + \frac{1}{2}\lambda = 6$$
$$\therefore\ \lambda = \frac{6}{7} \ \ldots\ldots\ldots(4)$$

Substitute the value of λ in (1),(2),(3) we obtain

$$x = \frac{6}{7} \quad , \quad y = \frac{9}{7} \quad , \quad z = \frac{-3}{7}$$

The nearest point is $p\left(\frac{6}{7}, \frac{9}{7}, \frac{-3}{7}\right)$.

Example 4: Find the extrema point for the function $f = 2x^2 + xy + y^2 - 2y$, subject to the constraint $2x - y = 1$, use Lagrange method.

Solution: Let $f = 2x^2 + xy + y^2 - 2y$

$$g = 2x - y - 1 = 0$$

$$\nabla f(x, y) = \lambda \ \nabla g(x, y)$$

$$\therefore \quad (4x + y)i + (x + 2y - 2)j = \lambda \ (2i - j)$$

$$4x + y = 2\lambda \ \ldots\ldots\ldots\ldots\ldots(1)$$

$$x + 2y - 2 = -\lambda \ \ldots\ldots\ldots(2)$$

$$8x + 2y = 4\lambda$$

$$-x - 2y + 2 = \lambda$$

$$7x + 2 = 5\lambda$$

$x = \frac{5\lambda - 2}{7}$

Substitute the value of x in Eq.(1) then

$4\left(\frac{5\lambda-2}{7}\right) + y = 2\lambda \quad \Rightarrow \quad y = 2\lambda - \frac{20\lambda-8}{7} \quad \Rightarrow \quad y = \frac{14\lambda-20\lambda+8}{7}$

$$y = \frac{-6\lambda + 8}{7}$$

Substitute the value of x and y in constraint $2x - y = 1$

$$2\left(\frac{5\lambda - 2}{7}\right) - \left(\frac{-6\lambda + 8}{7}\right) = 1 \quad \Rightarrow \frac{10\lambda - 4 + 6\lambda - 8}{7} = 1$$

$$\therefore \quad \lambda = \frac{19}{16}$$

Substitute the value of λ in x and y we have $x = \frac{9}{16}$, $y = \frac{1}{8}$

and the extrema point is p $\left(\frac{9}{16}, \frac{1}{8}\right)$.

Example 5: At what points on the circle $x^2 + y^2 = 1$ does $f = xy$ have an absolute maximum and what is the maximum ?

Solution: Let $f = xy$

$$g = x^2 + y^2 = 1$$

$$\nabla f(x,y) = \lambda \ \nabla g(x,y)$$

$$\therefore \quad yi + xj = \lambda \ (2xi + 2yj)$$

$$y = 2x\lambda \ \Rightarrow \ \lambda = \frac{y}{2x} \quad \text{.........(1)}$$

$$x = 2y\lambda \ \Rightarrow \ \lambda = \frac{x}{2y} \quad \text{.........(2)}$$

From Eq. (1) and Eq. (2) we get $\frac{y}{2x} = \frac{x}{2y} \Rightarrow \ 2x^2 = 2y^2$

$$\Rightarrow \quad x^2 = y^2 \quad \text{.........(3)}$$

Substitute (3) in constraint $g = x^2 + y^2 = 1$

$$x^2 + x^2 = 1 \quad \Rightarrow \ 2x^2 = 1 \quad \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

If $x = \frac{1}{\sqrt{2}}$, Then $y^2 = x^2 \Rightarrow y^2 = (\frac{1}{\sqrt{2}})^2 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{2}}$

and

If $x = -\frac{1}{\sqrt{2}}$,then $y^2 = x^2 \Rightarrow y^2 = (-\frac{1}{\sqrt{2}})^2 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{2}}$

The extrema points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f(x,y) = xy$

at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Rightarrow f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$

at the point $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \Rightarrow f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

at the point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Rightarrow f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

at the point $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \Rightarrow f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

the function $f(x,y) = xy$ have two maximum points at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, the maximum value is $\frac{1}{2}$.

Example 6: Find the extrema point for the function $f = 49 - x^2 - y^2$, on the line $x + 3y = 10$, use Lagrange method.

Solution: Let $f = 49 - x^2 - y^2$
$g = x + 3y - 10 = 0$

$$\nabla f(x,y) = \lambda \ \nabla g(x,y)$$
$$\therefore \quad -2xi - 2yj = \lambda \ (i + 3j)$$

$$-2x = \lambda \ \Rightarrow \ \lambda = -2x \ \ldots\ldots\ldots(1)$$
$$-2y = 3\lambda \ \Rightarrow \ \lambda = -\frac{2}{3}y \ \ldots\ldots(2)$$

From (1) and (2) we get $-2x = -\frac{2}{3}y \Rightarrow x = \frac{1}{3}y$ …….(3)
Substitute the value of x in constraint $x + 3y = 10$
$\frac{1}{3}y + 3y = 10 \ \Rightarrow \ \frac{10}{3}y = 10 \ \Rightarrow \ y = 3$ and $x = 1$
and the extrema point is p(1,3).

Example 7: Use Lagrange method to find the maximum value of $f = xy$, subject to the constraint $x + y - 16 = 0$.

Solution: Let $f = xy$
$g = x + y - 16 = 0$

$$\nabla f(x,y) = \lambda \ \nabla g(x,y)$$
$$\therefore \quad yi + xj = \lambda \ (i + j)$$

$$y = \lambda \ \ldots\ldots(1)$$
$$x = \lambda \ \ldots\ldots(2)$$

From (1) and (2) we get $x = y$ …….(3)
Substitute the value of x in constraint $x + y - 16 = 0$
$x + x = 16 \ \Rightarrow \ x = 8 \ \Rightarrow \ y = 8$
and the extrema point is p(8,8).

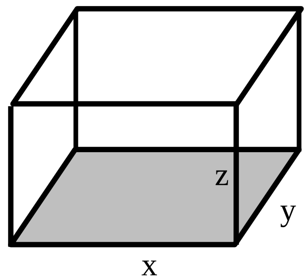
Example 8: Determine the dimensions(ابعاد) of a rectangular box, the sum of edges (اضلاعه) is 120cm and requiring the maximum volume of this box.

Solution: Frist method

Let $v = xyz$ …..(*) (volume of rectangular box) and

$x + y + z = 120$ (given) $\Rightarrow$ $z = 120 - x - y$

Substitute the value of z in Eq. (*)



$v = xy(120 - x - y)$
$v = 120xy - x^2y - xy^2$

$v_x = 120y - 2xy - y^2 = 0$ …….(1)
$v_y = 120x - x^2 - 2xy = 0$ …….(2)
From Eq (1): $y(120 - 2x - y) = 0$

Neither $y = 0$ or $y = 120 - 2x$ …….(3)

from Eq (2) , we obtain: $x(120 - x - 2y) = 0$
Neither $x = 0$ or $x = 120 - 2y$…..(4)

Substitute the value of x in Eq. (3)
Then $y = 120 - 2(120 - 2y) \Rightarrow y = 120 - 240 + 4y \Rightarrow -3y = -120$
$\Rightarrow$ $y = 40$, $\Rightarrow$ $x = 40$
From $z = 120 - x - y$ then $z = 40$, $\therefore v = xyz = 40.40.40 = 64000$

Second method (Lagrange method)

Let $v = xyz$ and $g = x + y + z - 120$

$$\nabla f(x, y) = \lambda \ \nabla g(x, y)$$
$$\therefore \quad yzi + xzj + xyk = \lambda \ (i + j + k)$$

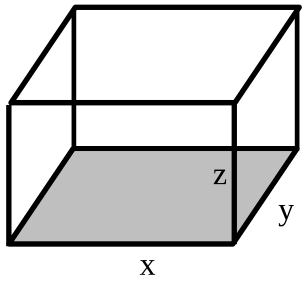
$$yz = \lambda \text{ ……(1)}$$
$$xz = \lambda \text{ ……(2)}$$
$$xy = \lambda \text{ ……(3)}$$

From (1) and (2) we get $yz = xz$ $\Rightarrow$ $y = x$ …….(4)
From (1) and (3) we get $yz = xy$ $\Rightarrow$ $z = x$ …….(5)

Substitute the Eg.(4) and (5) in constraint $x + y + z - 120 = 0$
$x + x + x = 120$ $\Rightarrow$ $3x = 120$ $\Rightarrow$ $x = 40$ therefore and $y = 40$
$z = 40$.

Example 9: Determine the dimensions of a rectangular box open at the top (بدون غطاء), having a volume of (32) and requiring the least amount (اقل كمية) of material (مواد) for its construction (لتكوينه).

Solution: Frist method



The volume of rectangular box equal to 32

i.e v = 32 $\Rrightarrow$ $xyz = 32 \Rrightarrow z = \frac{32}{xy}$ ……(*)

And f = xy + 2xz + 2yz

Substitute the value of z in above equation

$f = xy + 2x\left(\frac{32}{xy}\right) + 2y\left(\frac{32}{xy}\right) \Rrightarrow f = xy + \frac{64}{y} + \frac{64}{x}$

$f_x = y - \frac{64}{x^2} = 0$ …….(1)

$f_y = x - \frac{64}{y^2} = 0$ …….(2)

From Eq (2) $x = \frac{64}{y^2}$ ……..(3)

Substitute the value of x in Eq.(1) we obtain

$$y - \frac{64}{\left(\frac{64}{y^2}\right)^2} = 0 \Rrightarrow y = \frac{64}{\left(\frac{64}{y^2}\right)^2} \Rrightarrow y = \frac{1}{\frac{64}{y^4}} \Rrightarrow y\left(\frac{64}{y^4}\right) = 1 \Rrightarrow y^3 = 64$$

$\therefore y = 4$, substitute the value of y in Eq.(3) we obtain $x = 4$

Now, substitute the value of x and y in Eq.(*) we obtain $z = 2$

So, the dimensions of rectangular box are $x = 4$, $y = 4$, $z = 2$.

Second method (Lagrange method)

Let f = xy + 2xz + 2yz and g = xyz − 32

$$\nabla f(x,y,z) = \lambda\ \nabla g(x,y,z)$$

$$\therefore \quad (y+2z)i + (x+2z)j + (2x+2y)k = \lambda\ (yzi + xzj + xyk)$$

$$y + 2z = yz\lambda \quad \Rrightarrow \lambda = \frac{y+2z}{yz} \text{ ……(1)}$$

$$x + 2z = xz\lambda \quad \Rrightarrow \lambda = \frac{x+2z}{xz} \text{ ……(2)}$$

$$2x + 2y = xy\lambda \quad \Rrightarrow \lambda = \frac{2x+2y}{xy} \text{ ……(3)}$$

From (1) and (2) we get $\frac{y+2z}{yz} = \frac{x+2z}{xz} \Rrightarrow (y+2z)xz = (x+2z)yz$

$xy + 2xz = xy + 2yz \Rrightarrow xz = yz \Rrightarrow x = y$ ……(4)

From (1) and (3) we get $\frac{y+2z}{yz} = \frac{2x+2y}{xy} \Rrightarrow (y+2z)xy = (2x+2y)yz$

$xy + 2xz = 2xz + 2yz \Rrightarrow xy = 2yz \Rrightarrow x = 2z \Rrightarrow z = \frac{x}{2}$ ….(5)

Substitute the Eq. (4)and (5)in constraint xyz − 32 = 0

$x\,x\frac{x}{2} = 32 \Rrightarrow x^3 = 64 \Rrightarrow x = 4$, $y = 4$, $z = 2$