Solution

As is apparent from the diagram, the relationship between x and θ is given by $x = a \cdot \tan\theta$, so that

$$\frac{d\theta}{dx} = \frac{a}{a^2+x^2}$$

and it follows that

$$g(x) = \frac{1}{\pi} \cdot \left| \frac{a}{a^2+x^2} \right|$$
$$= \frac{1}{\pi} \cdot \frac{a}{a^2+x^2} \quad \text{for} -\infty < x < \infty$$

according to Theorem 1.

EXAMPLE 8

If $F(x)$ is the value of the distribution function of the continuous random variable X at x, find the probability density of $Y = F(X)$.

Solution

As can be seen from Figure 6, the value of Y corresponding to any particular value of X is given by the area under the curve, that is, the area under the graph of the density of X to the left of x. Differentiating $y = F(x)$ with respect to x, we get

$$\frac{dy}{dx} = F'(x) = f(x)$$

and hence

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f(x)}$$

provided $f(x) \neq 0$. It follows from Theorem 1 that

$$g(y) = f(x) \cdot \left| \frac{1}{f(x)} \right| = 1$$

for $0 < y < 1$, and we can say that y has the uniform density with $\alpha = 0$ and $\beta = 1$.

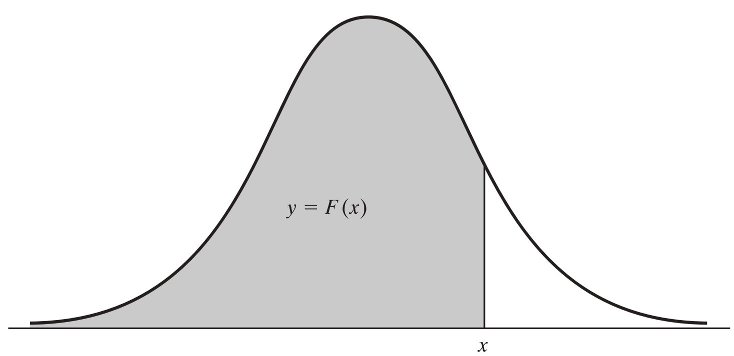

Figure 6. Diagram for Example 8.

The transformation that we performed in this example is called the **probability integral transformation**. Not only is the result of theoretical importance, but it facilitates the **simulation** of observed values of continuous random variables. A reference to how this is done, especially in connection with the normal distribution, is given in the end of the chapter.

When the conditions underlying Theorem 1 are not met, we can be in serious difficulties, and we may have to use the method of Section 2 or a generalization of Theorem 1 referred to among the references at the end of the chapter; sometimes, there is an easy way out, as in the following example.

EXAMPLE 9

If X has the standard normal distribution, find the probability density of $Z = X^2$.

Solution

Since the function given by $z = x^2$ is decreasing for negative values of x and increasing for positive values of x, the conditions of Theorem 1 are not met. However, the transformation from X to Z can be made in two steps: First, we find the probability density of $Y = |X|$, and then we find the probability density of $Z = Y^2 (= X^2)$.

As far as the first step is concerned, we already studied the transformation $Y = |X|$ in Example 2; in fact, we showed that if X has the standard normal distribution, then $Y = |X|$ has the probability density

$$g(y) = 2n(y; 0, 1) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

for $y > 0$, and $g(y) = 0$ elsewhere. For the second step, the function given by $z = y^2$ is increasing for $y > 0$, that is, for all values of Y for which $g(y) \neq 0$. Thus, we can use Theorem 1, and since

$$\frac{dy}{dz} = \frac{1}{2} z^{-\frac{1}{2}}$$

we get

$$\begin{aligned} h(z) &= \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z} \left| \frac{1}{2} z^{-\frac{1}{2}} \right| \\ &= \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{1}{2}z} \end{aligned}$$

for $z > 0$, and $h(z) = 0$ elsewhere. Observe that since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the distribution we have arrived at for Z is a chi-square distribution with $\nu = 1$.

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4 Transformation Technique: Several Variables

The method of the preceding section can also be used to find the distribution of a random variable that is a function of two or more random variables. Suppose, for instance, that we are given the joint distribution of two random variables X_1 and X_2 and that we want to determine the probability distribution or the probability density of the random variable $Y = u(X_1, X_2)$. If the relationship between y and x_1 with x_2 held constant or the relationship between y and x_2 with x_1 held constant permits, we can proceed in the discrete case as in Example 4 to find the joint distribution of Y and X_2 or that of X_1 and Y and then sum on the values of the other random variable to get the marginal distribution of Y. In the continuous case, we first use Theorem 1 with the transformation formula written as

$$g(y, x_2) = f(x_1, x_2) \cdot \left| \frac{\partial x_1}{\partial y} \right|$$

or as

$$g(x_1, y) = f(x_1, x_2) \cdot \left| \frac{\partial x_2}{\partial y} \right|$$

where $f(x_1, x_2)$ and the partial derivative must be expressed in terms of y and x_2 or x_1 and y. Then we integrate out the other variable to get the marginal density of Y.

EXAMPLE 10

If X_1 and X_2 are independent random variables having Poisson distributions with the parameters λ_1 and λ_2, find the probability distribution of the random variable $Y = X_1 + X_2$.

Solution

Since X_1 and X_2 are independent, their joint distribution is given by

$$\begin{aligned} f(x_1, x_2) &= \frac{e^{-\lambda_1}(\lambda_1)^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2}(\lambda_2)^{x_2}}{x_2!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1)^{x_1}(\lambda_2)^{x_2}}{x_1!x_2!} \end{aligned}$$

for $x_1 = 0, 1, 2, \ldots$ and $x_2 = 0, 1, 2, \ldots$. Since $y = x_1 + x_2$ and hence $x_1 = y - x_2$, we can substitute $y - x_2$ for x_1, getting

$$g(y, x_2) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}}{x_2!(y-x_2)!}$$

for $y = 0, 1, 2, \ldots$ and $x_2 = 0, 1, \ldots, y$, for the joint distribution of Y and X_2. Then, summing on x_2 from 0 to y, we get

$$\begin{aligned} h(y) &= \sum_{x_2=0}^{y} \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2}}{x_2!(y-x_2)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} \cdot \sum_{x_2=0}^{y} \frac{y!}{x_2!(y-x_2)!}(\lambda_2)^{x_2}(\lambda_1)^{y-x_2} \end{aligned}$$

after factoring out $e^{-(\lambda_1+\lambda_2)}$ and multiplying and dividing by $y!$. Identifying the summation at which we arrived as the binomial expansion of $(\lambda_1+\lambda_2)^y$, we finally get

$$h(y) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^y}{y!} \quad \text{for } y = 0, 1, 2, \ldots$$

and we have thus shown that the sum of two independent random variables having Poisson distributions with the parameters λ_1 and λ_2 has a Poisson distribution with the parameter $\lambda = \lambda_1 + \lambda_2$.

EXAMPLE 11

If the joint probability density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & \text{for } x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of $Y = \dfrac{X_1}{X_1+X_2}$.

Solution

Since y decreases when x_2 increases and x_1 is held constant, we can use Theorem 1 to find the joint density of X_1 and Y. Since $y = \dfrac{x_1}{x_1+x_2}$ yields $x_2 = x_1 \cdot \dfrac{1-y}{y}$ and hence

$$\frac{\partial x_2}{\partial y} = -\frac{x_1}{y^2}$$

it follows that

$$g(x_1, y) = e^{-x_1/y}\left|-\frac{x_1}{y^2}\right| = \frac{x_1}{y^2} \cdot e^{-x_1/y}$$

for $x_1 > 0$ and $0 < y < 1$. Finally, integrating out x_1 and changing the variable of integration to $u = x_1/y$, we get

$$\begin{aligned} h(y) &= \int_0^\infty \frac{x_1}{y^2} \cdot e^{-x_1/y} dx_1 \\ &= \int_0^\infty u \cdot e^{-u} du \\ &= \Gamma(2) \\ &= 1 \end{aligned}$$

for $0 < y < 1$, and $h(y) = 0$ elsewhere. Thus, the random variable Y has the uniform density with $\alpha = 0$ and $\beta = 1$. (Note that in Exercise 7 the reader was asked to show this by the distribution function technique.)

The preceding example could also have been worked by a general method where we begin with the joint distribution of two random variables X_1 and X_2 and determine the joint distribution of two new random variables $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$. Then we can find the marginal distribution of Y_1 or Y_2 by summation or integration.

This method is used mainly in the continuous case, where we need the following theorem, which is a direct generalization of Theorem 1.

THEOREM 2. Let $f(x_1, x_2)$ be the value of the joint probability density of the continuous random variables X_1 and X_2 at (x_1, x_2). If the functions given by $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ are partially differentiable with respect to both x_1 and x_2 and represent a one-to-one transformation for all values within the range of X_1 and X_2 for which $f(x_1, x_2) \neq 0$, then, for these values of x_1 and x_2, the equations $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 to give $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$, and for the corresponding values of y_1 and y_2, the joint probability density of $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ is given by

$$g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)] \cdot |J|$$

Here, J, called the **Jacobian** of the transformation, is the determinant

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Elsewhere, $g(y_1, y_2) = 0$.

We shall not prove this theorem, but information about Jacobians and their applications can be found in most textbooks on advanced calculus. There they are used mainly in connection with multiple integrals, say, when we want to change from rectangular coordinates to polar coordinates or from rectangular coordinates to spherical coordinates.

EXAMPLE 12

With reference to the random variables X_1 and X_2 of Example 11, find

(a) the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_1}{X_1 + X_2}$;

(b) the marginal density of Y_2.

Solution

(a) Solving $y_1 = x_1 + x_2$ and $y_2 = \dfrac{x_1}{x_1 + x_2}$ for x_1 and x_2, we get $x_1 = y_1 y_2$ and $x_2 = y_1(1 - y_2)$, and it follows that

$$J = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1$$

Since the transformation is one-to-one, mapping the region $x_1 > 0$ and $x_2 > 0$ in the x_1x_2-plane into the region $y_1 > 0$ and $0 < y_2 < 1$ in the y_1y_2-plane, we can use Theorem 2 and it follows that

$$g(y_1, y_2) = e^{-y_1}|-y_1| = y_1e^{-y_1}$$

for $y_1 > 0$ and $0 < y_2 < 1$; elsewhere, $g(y_1, y_2) = 0$.

(b) Using the joint density obtained in part (a) and integrating out y_1, we get

$$\begin{aligned} h(y_2) &= \int_0^\infty g(y_1, y_2)\, dy_1 \\ &= \int_0^\infty y_1e^{-y_1}\, dy_1 \\ &= \Gamma(2) \\ &= 1 \end{aligned}$$

for $0 < y_2 < 1$; elsewhere, $h(y_2) = 0$.

EXAMPLE 13

If the joint density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 1 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find

(a) the joint density of $Y = X_1 + X_2$ and $Z = X_2$;

(b) the marginal density of Y.

Note that in Exercise 6 the reader was asked to work the same problem by the distribution function technique.

Solution

(a) Solving $y = x_1 + x_2$ and $z = x_2$ for x_1 and x_2, we get $x_1 = y - z$ and $x_2 = z$, so that

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Because this transformation is one-to-one, mapping the region $0 < x_1 < 1$ and $0 < x_2 < 1$ in the x_1x_2-plane into the region $z < y < z + 1$ and $0 < z < 1$ in the yz-plane (see Figure 7), we can use Theorem 2 and we get

$$g(y, z) = 1 \cdot |1| = 1$$

for $z < y < z + 1$ and $0 < z < 1$; elsewhere, $g(y, z) = 0$.

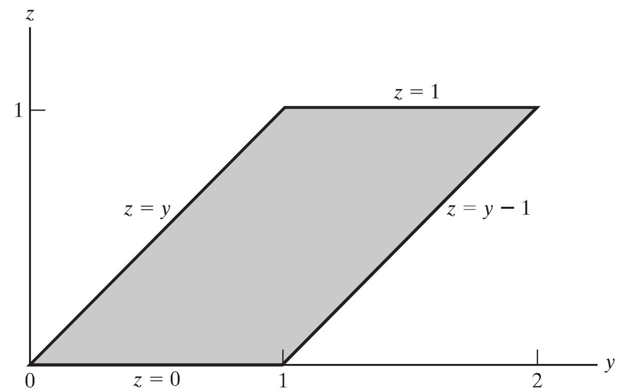

Figure 7. Transformed sample space for Example 13.

(b) Integrating out z separately for $y \leqq 0$, $0 < y < 1$, $1 < y < 2$, and $y \geqq 2$, we get

$$h(y) = \begin{cases} 0 & \text{for } y \leqq 0 \\ \displaystyle\int_0^y 1 \cdot dz = y & \text{for } 0 < y < 1 \\ \displaystyle\int_{y-1}^1 1 \cdot dz = 2 - y & \text{for } 1 < y < 2 \\ 0 & \text{for } y \geqq 2 \end{cases}$$

and to make the density function continuous, we let $h(1) = 1$. We have thus shown that the sum of the given random variables has the **triangular probability density** whose graph is shown in Figure 8.

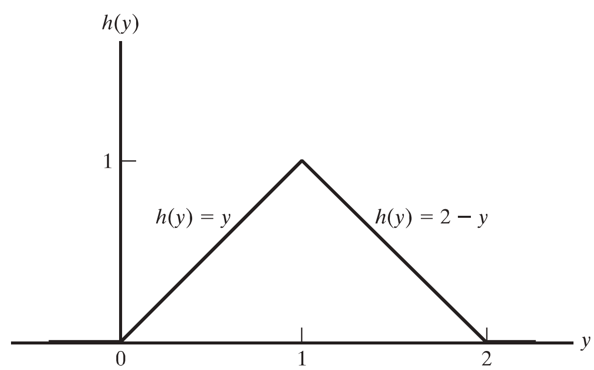

Figure 8. Triangular probability density.

So far we have considered here only functions of two random variables, but the method based on Theorem 2 can easily be generalized to functions of three or more random variables. For instance, if we are given the joint probability density of three random variables X_1, X_2, and X_3 and we want to find the joint probability density of the random variables $Y_1 = u_1(X_1, X_2, X_3)$, $Y_2 = u_2(X_1, X_2, X_3)$, and

$Y_3 = u_3(X_1, X_2, X_3)$, the general approach is the same, but the Jacobian is now the 3×3 determinant

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} & \dfrac{\partial x_1}{\partial y_3} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} & \dfrac{\partial x_2}{\partial y_3} \\ \dfrac{\partial x_3}{\partial y_1} & \dfrac{\partial x_3}{\partial y_2} & \dfrac{\partial x_3}{\partial y_3} \end{vmatrix}$$

Once we have determined the joint probability density of the three new random variables, we can find the marginal density of any two of the random variables, or any one, by integration.

EXAMPLE 14

If the joint probability density of X_1, X_2, and X_3 is given by

$$f(x_1, x_2, x_3) = \begin{cases} e^{-(x_1+x_2+x_3)} & \text{for } x_1 > 0, x_2 > 0, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find

(a) the joint density of $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_2$, and $Y_3 = X_3$;

(b) the marginal density of Y_1.

Solution

(a) Solving the system of equations $y_1 = x_1 + x_2 + x_3$, $y_2 = x_2$, and $y_3 = x_3$ for x_1, x_2, and x_3, we get $x_1 = y_1 - y_2 - y_3$, $x_2 = y_2$, and $x_3 = y_3$. It follows that

$$J = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

and, since the transformation is one-to-one, that

$$\begin{aligned} g(y_1, y_2, y_3) &= e^{-y_1} \cdot |1| \\ &= e^{-y_1} \end{aligned}$$

for $y_2 > 0$, $y_3 > 0$, and $y_1 > y_2 + y_3$; elsewhere, $g(y_1, y_2, y_3) = 0$.

(b) Integrating out y_2 and y_3, we get

$$\begin{aligned} h(y_1) &= \int_0^{y_1} \int_0^{y_1 - y_3} e^{-y_1}\, dy_2\, dy_3 \\ &= \frac{1}{2} y_1^2 \cdot e^{-y_1} \end{aligned}$$

for $y_1 > 0$; $h(y_1) = 0$ elsewhere. Observe that we have shown that the sum of three independent random variables having the gamma distribution with $\alpha = 1$ and $\beta = 1$ is a random variable having the gamma distribution with $\alpha = 3$ and $\beta = 1$.

As the reader will find in Exercise 39, it would have been easier to obtain the result of part (b) of Example 14 by using the method based on Theorem 1.

Exercises

9. If X has a hypergeometric distribution with $M = 3$, $N = 6$, and $n = 2$, find the probability distribution of Y, the number of successes minus the number of failures.

10. With reference to Exercise 9, find the probability distribution of the random variable $Z = (X-1)^2$.

11. If X has a binomial distribution with $n = 3$ and $\theta = \frac{1}{3}$, find the probability distributions of
(a) $Y = \dfrac{X}{1+X}$;
(b) $U = (X-1)^4$.

12. If X has a geometric distribution with $\theta = \frac{1}{3}$, find the formula for the probability distribution of the random variable $Y = 4-5X$.

13. This question has been intentionally omitted for this edition.

14. This question has been intentionally omitted for this edition.

15. Use the transformation technique to rework Exercise 2.

16. If the probability density of X is given by

$$f(x) = \begin{cases} \dfrac{kx^3}{(1+2x)^6} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where k is an appropriate constant, find the probability density of the random variable $Y = \dfrac{2X}{1+2X}$. Identify the distribution of Y, and thus determine the value of k.

17. If the probability density of X is given by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability density of $Y = X^3$. Also, plot the graphs of the probability densities of X and Y and indicate the respective areas under the curves that represent $P(\frac{1}{2} < X < 1)$ and $P(\frac{1}{8} < Y < 1)$.

18. If X has a uniform density with $\alpha = 0$ and $\beta = 1$, show that the random variable $Y = -2$. In X has a gamma distribution. What are its parameters?

19. This question has been intentionally omitted for this edition.

20. Consider the random variable X with the probability density

$$f(x) = \begin{cases} \dfrac{3x^2}{2} & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Use the result of Example 2 to find the probability density of $Y = |X|$.
(b) Find the probability density of $Z = X^2 (= Y^2)$.

21. Consider the random variable X with the uniform density having $\alpha = 1$ and $\beta = 3$.
(a) Use the result of Example 2 to find the probability density of $Y = |X|$.
(b) Find the probability density of $Z = X^4 (= Y^4)$.

22. If the joint probability distribution of X_1 and X_2 is given by

$$f(x_1, x_2) = \frac{x_1 x_2}{36}$$

for $x_1 = 1, 2, 3$ and $x_2 = 1, 2, 3$, find
(a) the probability distribution of $X_1 X_2$;
(b) the probability distribution of X_1/X_2.

23. With reference to Exercise 22, find
(a) the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$;
(b) the marginal distribution of Y_1.

24. If the joint probability distribution of X and Y is given by

$$f(x, y) = \frac{(x-y)^2}{7}$$

for $x = 1, 2$ and $y = 1, 2, 3$, find
(a) the joint distribution of $U = X + Y$ and $V = X - Y$;
(b) the marginal distribution of U.

25. If X_1, X_2, and X_3 have the multinomial distribution with $n = 2$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{1}{3}$, and $\theta_3 = \frac{5}{12}$, find the joint probability distribution of $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$, and $Y_3 = X_3$.

26. This question has been intentionally omitted for this edition.