27. If X_1 and X_2 are independent random variables having binomial distributions with the respective parameters n_1 and θ and n_2 and θ, show that $Y = X_1 + X_2$ has the binomial distribution with the parameters $n_1 + n_2$ and θ.

28. If X_1 and X_2 are independent random variables having the geometric distribution with the parameter θ, show that $Y = X_1 + X_2$ is a random variable having the negative binomial distribution with the parameters θ and $k = 2$.

29. If X and Y are independent random variables having the standard normal distribution, show that the random variable $Z = X + Y$ is also normally distributed. (*Hint*: Complete the square in the exponent.) What are the mean and the variance of this normal distribution?

30. Consider two random variables X and Y with the joint probability density

$$f(x,y) = \begin{cases} 12xy(1-y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability density of $Z = XY^2$ by using Theorem 1 to determine the joint probability density of Y and Z and then integrating out y.

31. Rework Exercise 30 by using Theorem 2 to determine the joint probability density of $Z = XY^2$ and $U = Y$ and then finding the marginal density of Z.

32. Consider two independent random variables X_1 and X_2 having the same Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for } -\infty < x < \infty$$

Find the probability density of $Y_1 = X_1 + X_2$ by using Theorem 1 to determine the joint probability density of X_1 and Y_1 and then integrating out x_1. Also, identify the distribution of Y_1.

33. Rework Exercise 32 by using Theorem 2 to determine the joint probability density of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ and then finding the marginal density of Y_1.

34. Consider two random variables X and Y whose joint probability density is given by

$$f(x,y) = \begin{cases} \frac{1}{2} & \text{for } x > 0, y > 0, x + y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability density of $U = Y - X$ by using Theorem 1.

35. Rework Exercise 34 by using Theorem 2 to determine the joint probability density of $U = Y - X$ and $V = X$ and then finding the marginal density of U.

36. Let X_1 and X_2 be two continuous random variables having the joint probability density

$$f(x_1,x_2) = \begin{cases} 4x_1x_2 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint probability density of $Y_1 = X_1^2$ and $Y_2 = X_1X_2$.

37. Let X and Y be two continuous random variables having the joint probability density

$$f(x,y) = \begin{cases} 24xy & \text{for } 0 < x < 1, 0 < y < 1, x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint probability density of $Z = X + Y$ and $W = X$.

38. Let X and Y be two independent random variables having identical gamma distributions.
(a) Find the joint probability density of the random variables $U = \dfrac{X}{X+Y}$ and $V = X + Y$.
(b) Find and identify the marginal density of U.

39. The method of transformation based on Theorem 1 can be generalized so that it applies also to random variables that are functions of two or more random variables. So far we have used this method only for functions of two random variables, but when there are three, for example, we introduce the new random variable in place of one of the original random variables, and then we eliminate (by summation or integration) the other two random variables with which we began. Use this method to rework Example 14.

40. In Example 13 we found the probability density of the sum of two independent random variables having the uniform density with $\alpha = 0$ and $\beta = 1$. Given a third random variable X_3, which has the same uniform density and is independent of both X_1 and X_2, show that if $U = Y + X_3 = X_1 + X_2 + X_3$, then
(a) the joint probability density of U and Y is given by

$$g(u,y) = \begin{cases} y & \text{for Regions I and II of Figure 9} \\ 2-y & \text{for Regions III and IV of Figure 9} \\ 0 & \text{elsewhere} \end{cases}$$

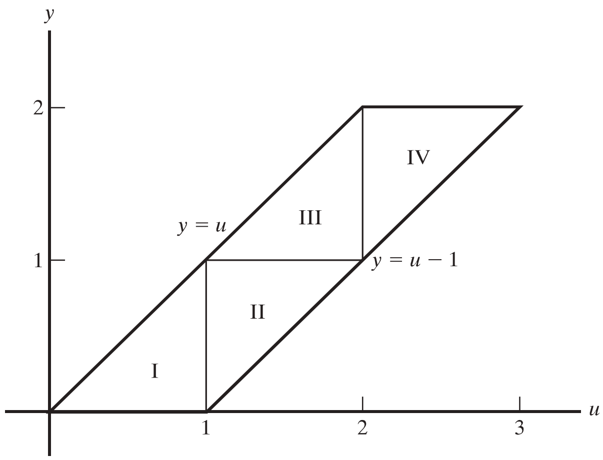

Figure 9. Diagram for Exercise 40.

(b) the probability density of U is given by

$$h(u) = \begin{cases} 0 & \text{for } u \leqq 0 \\ \frac{1}{2}u^2 & \text{for } 0 < u < 1 \\ \frac{1}{2}u^2 - \frac{3}{2}(u-1)^2 & \text{for } 1 < u < 2 \\ \frac{1}{2}u^2 - \frac{3}{2}(u-1)^2 + \frac{3}{2}(u-2)^2 & \text{for } 2 < u < 3 \\ 0 & \text{for } u \geqq 3 \end{cases}$$

Note that if we let $h(1) = h(2) = \frac{1}{2}$, this will make the probability density of U continuous.

5 Moment-Generating Function Technique

Moment-generating functions can play an important role in determining the probability distribution or density of a function of random variables when the function is a linear combination of n *independent* random variables. We shall illustrate this technique here when such a linear combination is, in fact, the sum of n independent random variables, leaving it to the reader to generalize it in Exercises 45 and 46.

The method is based on the following theorem that the moment-generating function of the sum of n independent random variables equals the product of their moment-generating functions.

THEOREM 3. If $X_1, X_2, \ldots$, and X_n are independent random variables and $Y = X_1 + X_2 + \cdots + X_n$, then

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

where $M_{X_i}(t)$ is the value of the moment-generating function of X_i at t.

Proof Making use of the fact that the random variables are independent and hence

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) \cdot f_2(x_2) \cdot \ldots \cdot f_n(x_n)$$

according to the following definition "**INDEPENDENCE OF DISCRETE RANDOM VARIABLES**. *If* $f(x_1, x_2, \ldots, x_n)$ *is the value of the joint probability distribution of the discrete random variables* $X_1, X_2, \ldots, X_n$ *at* $(x_1, x_2, \ldots, x_n)$ *and* $f_i(x_i)$ *is the value of the marginal distribution of* X_i *at* x_i *for* $i = 1, 2, \ldots, n$*, then the* n *random variables are* ***independent*** *if and only if* $f(x_1, x_2, \ldots, x_n) = f_1(x_1) \cdot f_2(x_2) \cdot \ldots \cdot f_n(x_n)$ *for all* $(x_1, x_2, \ldots, x_n)$ *within their range*", we can write

$$M_Y(t) = E(e^{Yt})$$
$$= E\left[e^{(X_1+X_2+\cdots+X_n)t}\right]$$
$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{(x_1+x_2+\cdots+x_n)t} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n$$
$$= \int_{-\infty}^{\infty} e^{x_1 t} f_1(x_1)\, dx_1 \cdot \int_{-\infty}^{\infty} e^{x_2 t} f_2(x_2)\, dx_2 \cdots \int_{-\infty}^{\infty} e^{x_n t} f_n(x_n)\, dx_n$$
$$= \prod_{i=1}^{n} M_{X_i}(t)$$

which proves the theorem for the continuous case. To prove it for the discrete case, we have only to replace all the integrals by sums.

Note that if we want to use Theorem 3 to find the probability distribution or the probability density of the random variable $Y = X_1 + X_2 + \cdots + X_n$, we must be able to identify whatever probability distribution or density corresponds to $M_Y(t)$.

EXAMPLE 15

Find the probability distribution of the sum of n independent random variables $X_1, X_2, \ldots, X_n$ having Poisson distributions with the respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Solution
By the theorem "The moment-generating function of the Poisson distribution is given by $M_X(t) = e^{\lambda(e^t-1)}$" we have

$$M_{X_i}(t) = e^{\lambda_i(e^t-1)}$$

hence, for $Y = X_1 + X_2 + \cdots + X_n$, we obtain

$$M_Y(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t-1)} = e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)(e^t-1)}$$

which can readily be identified as the moment-generating function of the Poisson distribution with the parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Thus, the distribution of the sum of n independent random variables having Poisson distributions with the parameters λ_i is a Poisson distribution with the parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Note that in Example 10 we proved this for $n = 2$.

EXAMPLE 16

If $X_1, X_2, \ldots, X_n$ are independent random variables having exponential distributions with the same parameter θ, find the probability density of the random variable $Y = X_1 + X_2 + \cdots + X_n$.

Solution
Since the exponential distribution is a gamma distribution with $\alpha = 1$ and $\beta = \theta$, we have

$$M_{X_i}(t) = (1 - \theta t)^{-1}$$

and hence

$$M_Y(t) = \prod_{i=1}^{n}(1-\theta t)^{-1} = (1-\theta t)^{-n}$$

Identifying the moment-generating function of Y as that of a gamma distribution with $\alpha = n$ and $\beta = \theta$, we conclude that the distribution of the sum of n independent random variables having exponential distributions with the same parameter θ is a gamma distribution with the parameters $\alpha = n$ and $\beta = \theta$. Note that this agrees with the result of Example 14, where we showed that the sum of three independent random variables having exponential distributions with the parameter $\theta = 1$ has a gamma distribution with $\alpha = 3$ and $\beta = 1$.

Theorem 3 also provides an easy and elegant way of deriving the moment-generating function of the binomial distribution. Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables having the same Bernoulli distribution $f(x;\theta) = \theta^x(1-\theta)^{1-x}$ for $x = 0, 1$. We have

$$M_{X_i}(t) = e^{0\cdot t}(1-\theta) + e^{1\cdot t}\theta = 1 + \theta(e^t - 1)$$

so that Theorem 3 yields

$$M_Y(t) = \prod_{i=1}^{n}[1+\theta(e^t-1)] = [1+\theta(e^t-1)]^n$$

This moment-generating function is readily identified as that of the binomial distribution with the parameters n and θ. Of course, $Y = X_1 + X_2 + \cdots + X_n$ is the total number of successes in n trials, since X_1 is the number of successes on the first trial, X_2 is the number of successes on the second trial, ..., and X_n is the number of successes on the nth trial. This is a fruitful way of looking at the binomial distribution.

Exercises

41. Use the moment-generating function technique to rework Exercise 27.

42. Find the moment-generating function of the negative binomial distribution by making use of the fact that if k independent random variables have geometric distributions with the same parameter θ, their sum is a random variable having the negative binomial distribution with the parameters θ and k.

43. If n independent random variables have the same gamma distribution with the parameters α and β, find the moment-generating function of their sum and, if possible, identify its distribution.

44. If n independent random variables X_i have normal distributions with the means μ_i and the standard deviations σ_i, find the moment-generating function of their sum and identify the corresponding distribution, its mean, and its variance.

45. Prove the following generalization of Theorem 3: If $X_1, X_2, \ldots$, and X_n are independent random variables and $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$, then

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

where $M_{X_i}(t)$ is the value of the moment-generating function of X_i at t.

46. Use the result of Exercise 45 to show that, if n independent random variables X_i have normal distributions with the means μ_i and the standard deviations σ_i, then $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ has a normal distribution. What are the mean and the variance of this distribution?

6 The Theory in Application

Examples of the need for transformations in solving practical problems abound. To illustrate these applications, we give three examples. The first example illustrates an application of the transformation technique to a simple problem in electrical engineering.

EXAMPLE 17

Suppose the resistance in a simple circuit varies randomly in response to environmental conditions. To determine the effect of this variation on the current flowing through the circuit, an experiment was performed in which the resistance (R) was varied with equal probabilities on the interval $0 < R \leq A$ and the ensuing voltage (E) was measured. Find the distribution of the random variable I, the current flowing through the circuit.

Solution

Using the well-known relation $E = IR$, we have $I = u(R) = \frac{E}{R}$. The probability distribution of R is given by $f(R) = \frac{1}{A}$ for $0 < R \leq A$. Thus, $w(I) = \frac{E}{I}$, and the probability density of I is given by

$$g(I) = f(R) \cdot |w'(I)| = \frac{1}{A}\left|-\frac{E}{R^2}\right| = \frac{E}{AR^2} \qquad R > 0$$

It should be noted, with respect to this example, that this is a designed experiment in as much as the distribution of R was preselected as a uniform distribution. If the nominal value of R is to be the mean of this distribution, some other distribution might have been selected to impart better properties to this estimate.

The next example illustrates transformations of data to normality.

EXAMPLE 18

What underlying distribution of the data is assumed when the square-root transformation is used to obtain approximately normally distributed data? (Assume the data are nonnegative, that is, the probability of a negative observation is zero.)

Solution

A simple alternate way to use the distribution-function technique is to write down the differential element of the density function, $f(x)\,dx$, of the transformed observations, y, and to substitute x^2 for y. (When we do this, we must remember that the differential element, dy, must be changed to $dx = 2x\,dx$.) We obtain

$$f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} \cdot 2x \cdot e^{-\frac{1}{2}(x^2-\mu)^2/\sigma^2}\,dx$$

The required density function is given by

$$f(x) = \sqrt{\frac{2}{\pi\sigma^2}}\,xe^{-\frac{1}{2}(x^2-\mu)^2/\sigma^2}$$

This distribution is not immediately recognizable, but it can be graphed quickly using appropriate computer software.

The final example illustrates an application to waiting-time problems.

EXAMPLE 19

Let us assume that the decay of a radioactive element is exponentially distributed, so that $f(x) = \lambda e^{-\lambda x}$ for $\lambda > 0$ and $x > 0$; that is, the time for the nucleus to emit the first α particle is x (in seconds). It can be shown that such a process has no memory; that is, the time *between successive emissions* also can be described by this distribution. Thus, it follows that successive emissions of α particles are independent. If the parameter λ equals 5, find the probability that a given substance will emit 2 particles in less than or equal to 3 seconds.

Solution

Let x_i be the waiting time between emissions i and $i+1$, for $i = 0, 1, 2, \ldots, n-1$. Then the total time for n emissions to take place is the sum $T = x_0 + x_1 + \cdots + x_{n-1}$. The moment-generating function of this sum is given in Example 16 to be

$$M_T(t) = (1 - t/\lambda)^{-n}$$

This can be recognized as the moment-generating function of the gamma distribution with parameters $\alpha = n = 2$ and $\beta = 1/\lambda = 1/5$. The required probability is given by

$$P\left(T \leq 3; \alpha = 10, \beta = \frac{1}{5}\right) = \frac{1}{\frac{1}{5}\Gamma(2)} \int_0^3 x\, e^{-5x} dx$$

Integrating by parts, the integral becomes

$$P(T \leq 3) = -\frac{1}{5} x e^{-5x}\Big|_0^3 - \int_0^3 -\frac{1}{5} e^{-5x}\, dx = 1 - 1.6e^{-15}$$

Without further evaluation, it is clear that this event is virtually certain to occur.

Applied Exercises SECS. 1–2

47. This question has been intentionally omitted for this edition.

48. This question has been intentionally omitted for this edition.

49. This question has been intentionally omitted for this edition.

50. Let X be the amount of premium gasoline (in 1,000 gallons) that a service station has in its tanks at the beginning of a day, and Y the amount that the service station sells during that day. If the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{200} & \text{for } 0 < y < x < 20 \\ 0 & \text{elsewhere} \end{cases}$$

use the distribution function technique to find the probability density of the amount that the service station has left in its tanks at the end of the day.

51. The percentages of copper and iron in a certain kind of ore are, respectively, X_1 and X_2. If the joint density of these two random variables is given by

$$f(x_1,x_2) = \begin{cases} \frac{3}{11}(5x_1 + x_2) & \text{for } x_1 > 0, x_2 > 0, \\ & \text{and } x_1 + 2x_2 < 2 \\ 0 & \text{elsewhere} \end{cases}$$

use the distribution function technique to find the probability density of $Y = X_1 + X_2$. Also find $E(Y)$, the expected total percentage of copper and iron in the ore.

SECS. 3–4

52. According to the Maxwell–Boltzmann law of theoretical physics, the probability density of V, the velocity of a gas molecule, is

$$f(v) = \begin{cases} kv^2 e^{-\beta v^2} & \text{for } v > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where β depends on its mass and the absolute temperature and k is an appropriate constant. Show that the kinetic energy $E = \frac{1}{2}mV^2$, where m the mass of the molecule is a random variable having a gamma distribution.

53. This question has been intentionally omitted for this edition.

54. This question has been intentionally omitted for this edition.

55. This question has been intentionally omitted for this edition.

56. Use a computer program to generate 10 "pseudorandom" numbers having the standard normal distribution.

57. Describe how the probability integral transformation might have been used by the writers of the software that you used to produce the result of Exercise 56.

SEC. 5

58. A lawyer has an unlisted number on which she receives on the average 2.1 calls every half-hour and a listed number on which she receives on the average 10.9 calls every half-hour. If it can be assumed that the numbers of calls that she receives on these phones are independent random variables having Poisson distributions, what are the probabilities that in half an hour she will receive altogether

(a) 14 calls;

(b) at most 6 calls?

59. In a newspaper ad, a car dealer lists a 2001 Chrysler, a 2010 Ford, and a 2008 Buick. If the numbers of inquiries he will get about these cars may be regarded as independent random variables having Poisson distributions with the parameters $\lambda_1 = 3.6$, $\lambda_2 = 5.8$, and $\lambda_3 = 4.6$, what are the probabilities that altogether he will receive

(a) fewer than 10 inquiries about these cars;

(b) anywhere from 15 to 20 inquiries about these cars;

(c) at least 18 inquiries about these cars?

60. With reference to Exercise 59, what is the probability that the car dealer will receive six inquiries about the 2010 Ford and eight inquiries about the other two cars?

61. If the number of complaints a dry-cleaning establishment receives per day is a random variable having the Poisson distribution with $\lambda = 3.3$, what are the probabilities that it will receive

(a) 2 complaints on any given day;

(b) 5 complaints altogether on any two given days;

(c) at least 12 complaints altogether on any three given days?

62. The number of fish that a person catches per hour at Woods Canyon Lake is a random variable having the Poisson distribution with $\lambda = 1.6$. What are the probabilities that a person fishing there will catch

(a) four fish in 2 hours;

(b) at least two fish in 3 hours;

(c) at most three fish in 4 hours?

63. If the number of minutes it takes a service station attendant to balance a tire is a random variable having an exponential distribution with the parameter $\theta = 5$, what are the probabilities that the attendant will take

(a) less than 8 minutes to balance two tires;

(b) at least 12 minutes to balance three tires?

64. If the number of minutes that a doctor spends with a patient is a random variable having an exponential distribution with the parameter $\theta = 9$, what are the probabilities that it will take the doctor at least 20 minutes to treat

(a) one patient; **(b)** two patients; **(c)** three patients?

65. If X is the number of 7's obtained when rolling a pair of dice three times, find the probability that $Y = X^2$ will exceed 2.

66. If X has the exponential distribution given by $f(x) = 0.5\,\mathrm{e}^{-0.5x}, x > 0$, find the probability that $x > 1$.

SEC. 6

67. If, d, the diameter of a circle is selected at random from the density function

$$f(d) = k\left(1 - \frac{d}{5}\right), 0 < d < 5,$$

(a) find the value of k so that *f(d)* is a probability density;

(b) find the density function of the areas of the circles so selected.

68. Show that the underlying distribution function of Example 18 is, indeed, a probability distribution, and use a computer program to graph the density function.

69. If $X = \ln Y$ has a normal distribution with the mean μ and the standard deviation σ, find the probability density of Y which is said to have the **log-normal** distribution.

70. The logarithm of the ratio of the output to the input current of a transistor is called its current gain. If current gain measurements made on a certain transistor are normally distributed with $\mu = 1.8$ and $\sigma = 0.05$, find the probability that the current gain will exceed the required minimum value of 6.0.

References

The use of the probability integral transformation in problems of simulation is discussed in

Johnson, R. A., *Miller and Freund's Probability and Statistics for Engineers*, 6th ed. Upper Saddle River, N.J.: Prentice Hall, 2000.

A generalization of Theorem 1, which applies when the interval within the range of X for which $f(x) \neq 0$ can be partitioned into k subintervals so that the conditions of Theorem 1 apply separately for each of the subintervals, may be found in

Walpole, R. E., and Myers, R. H., *Probability and Statistics for Engineers and Scientists*, 4th ed. New York: Macmillan Publishing Company, Inc., 1989.

More detailed and more advanced treatments of the material in this chapter are given in many advanced texts on mathematical statistics; for instance, in

Hogg, R. V., and Craig, A. T., *Introduction to Mathematical Statistics*, 4th ed. New York: Macmillan Publishing Company, Inc., 1978,

Roussas, G. G., *A First Course in Mathematical Statistics*. Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1973,

Wilks, S. S., *Mathematical Statistics*. New York: John Wiley & Sons, Inc., 1962.

Answers to Odd-Numbered Exercises

1 $g(y) = \frac{1}{\theta}e^y e^{-(1/\theta)}e^y$ for $-\infty < y < \infty$.

3 $g(y) = 2y$ for $0 < y < 1$ and $g(y) = 0$ elsewhere.

5 **(a)** $f(y) = \dfrac{1}{\theta_1 - \theta_2} \cdot (e^{-y/\theta_1} - e^{-y/\theta_2})$ for $y > 0$ and $f(y) = 0$ elsewhere; **(b)** $f(y) = \dfrac{1}{\theta^2} \cdot ye^{-y/\theta}$ for $y > 0$ and $f(y) = 0$ elsewhere.

9 $h(-2) = \frac{1}{5}, h(0) = \frac{3}{5}$, and $h(2) = \frac{1}{5}$.

11 **(a)** $g(0) = \frac{8}{27}$, $g(\frac{1}{2}) = \frac{12}{27}$, $g(\frac{2}{3}) = \frac{6}{27}$, $g(\frac{3}{4}) = \frac{1}{27}$; **(b)** $g(0) = \frac{12}{27}$, $g(1) = \frac{14}{27}$, $g(16) = \frac{1}{27}$.

13 $g(0) = \frac{1}{3}$, $g(1) = \frac{1}{3}$, $g(2) = \frac{1}{3}$.

17 $g(y) = \dfrac{1}{6}y^{\frac{-1}{3}}$.

21 **(a)** $g(y) = \frac{1}{8}y^{-3/4}$ for $0 < y < 1$ and $g(y) = \frac{1}{4}$ for $1 < y < 3$; **(b)** $h(z) = \frac{1}{16} \cdot z^{-3/4}$ for $1 < z < 81$ and $h(z) = 0$ elsewhere.

23 **(a)** $f(2,\ 0) = \frac{1}{36}$, $f(3,\ -1) = \frac{2}{36}$, $f(3,\ 1) = \frac{2}{36}$, $f(4,\ -2) = \frac{3}{36}$, $f(4,\ 0) = \frac{4}{36}$, $f(4,\ 2) = \frac{3}{36}$, $f(5,\ -1) = \frac{6}{36}$, $f(5,\ 1) = \frac{6}{36}$, and $f(6,\ 0) = \frac{9}{36}$; **(b)** $g(2) = \frac{1}{36}$, $g(3) = \frac{4}{36}$, $g(4) = \frac{10}{36}$, $g(5) = \frac{12}{36}$, and $g(6) = \frac{9}{36}$.

25 **(b)** $g(0,\ 0,\ 2) = \frac{25}{144}$, $g(1,\ -1,\ 1) = \frac{5}{18}$, $g(1,\ 1,\ 1) = \frac{5}{24}$, $g(2,\ -2,\ 0) = \frac{1}{9}$, $g(2,\ 0,\ 0) = \frac{1}{6}$, and $g(2,\ 2,\ 0) = \frac{1}{16}$.

29 $\mu = 0$ and $\sigma^2 = 2$.

31 $g(z,u) = 12z(u^{-3} - u^{-2})$ over the region bounded by $z = 0, u = 1$, and $z = u^2$, and $g(z,u) = 0$ elsewhere; $h(z) = 6z + 6 - 12\sqrt{z}$ for $0 < z < 1$ and $h(z) = 0$ elsewhere.

33 The marginal distribution is the Cauchy distribution $g(y) = \dfrac{1}{\pi} \cdot \dfrac{2}{4 + y^2}$ for $-\infty < y < \infty$.

35 $f(u,v) = \frac{1}{2}$ over the region bounded by $v = 0, u = -v$, and $2v + u = 2$, and $f(u,v) = 0$ elsewhere; $g(u) = \frac{1}{4}(2+u)$ for $-2 < u \leq 0, g(u) = \frac{1}{4}(2-u)$ for $0 < u < 2$ and $g(u) = 0$ elsewhere.

37 $g(w,\ z) = 24w(z-w)$ over the region bounded by $w = 0, z = 1$, and $z = w$; $g(w,z) = 0$ elsewhere.

43 It is a gamma distribution with the parameters αn and β.

51 $g(y) = \frac{9}{11} \cdot y^2$ for $0 < y \leq 1, g(y) = \dfrac{3(2-y)(7y-4)}{11}$ for $1 < y < 2$, and $g(y) = 0$ elsewhere.

53 $h(r) = 2r$ for $0 < r < 1$ and $h(r) = 0$ elsewhere.

55 $g(v,\ w) = 5e^{-v}$ for $0.2 < w < 0.4$ and $v > 0$; $h(v) = e^{-v}$ for $v > 0$ and $h(v) = 0$ elsewhere.

59 **(a)** 0.1093; **(b)** 0.3817; **(c)** 0.1728.

61 **(a)** 0.2008; **(b)** 0.1420; **(c)** 0.2919.

63 **(a)** 0.475; **(b)** 0.570.

65 $\frac{2}{27}$.

67 **(a)** $\frac{2}{5}$; **(b)** $g(A) = \frac{2}{5}\left(\dfrac{1}{\sqrt{\pi}}A^{-1/2} - 1\right)$ for $0 < A < \frac{25}{4}\pi$ and $g(A) = 0$ elsewhere.

69 $g(y) = \dfrac{1}{\sqrt{2\pi}\sigma} \cdot \dfrac{1}{y} \cdot e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}$ for $y > 0$ and $g(y) = 0$ elsewhere.

Sampling Distributions

1 Introduction

Statistics concerns itself mainly with conclusions and predictions resulting from chance outcomes that occur in carefully planned experiments or investigations. Drawing such conclusions usually involves taking sample observations from a given population and using the results of the sample to make inferences about the population itself, its mean, its variance, and so forth. To do this requires that we first find the distributions of certain functions of the random variables whose values make up the sample, called **statistics**. (An example of such a statistic is the sample mean.) The properties of these distributions then allow us to make probability statements about the resulting inferences drawn from the sample about the population. The theory to be given in this chapter forms an important foundation for the theory of statistical inference.

Inasmuch as statistical inference can be loosely defined as a process of drawing conclusions from a sample about the population from which it is drawn, it is useful to have the following definition.

> **DEFINITION 1. POPULATION.** *A set of numbers from which a sample is drawn is referred to as a* ***population****. The distribution of the numbers constituting a population is called the* ***population distribution****.*

To illustrate, suppose a scientist must choose and then weigh 5 of 40 guinea pigs as part of an experiment, a layman might say that the ones she selects constitute the sample. This is how the term "sample" is used in everyday language. In statistics, it is preferable to look upon the weights of the 5 guinea pigs as a sample from the population, which consists of the weights of all 40 guinea pigs. In this way, the population as well as the sample consists of numbers. Also, suppose that, to estimate the average useful life of a certain kind of transistor, an engineer selects 10 of these transistors, tests them over a period of time, and records for each one the time to failure. If these times to failure are values of independent random variables having an exponential distribution with the parameter θ, we say that they constitute a sample from this exponential population.

As can well be imagined, not all samples lend themselves to valid generalizations about the populations from which they came. In fact, most of the methods of inference discussed in this chapter are based on the assumption that we are dealing with

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