Mathematical Foundation (2)

1. Cardinal Numbers.
2. The Natural Numbers.
3. The Integers Numbers.
4. The Rational Numbers and the Real Numbers.
5. The Complex Numbers.

References:

1. Charles C. Pinter, (1971) "Set Theory".

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"مقدمة في أسس الرياضيات"

Cardinal Numbers

Definition:- let A and B are sets, then we said that the set A is equivalente to the set B if there exists a mapping $f : A \to B$ which is a bijective function and denoted by $(A \sim B)$ or $(A \equiv B)$.

Notes: 1. $A \equiv A$.

2. If $A \equiv B$, then $B \equiv A$.

3. If $A \equiv B$ and $B \equiv C$, then $A \equiv C$.

زائد

Def.:- A cardinal number of the set A is the number of elements in A and use the symbol ($\#(A)$ or $card(A)$).

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Note:- If A is a set and $\#(A) = \alpha$ and $\#(A) = \beta$, then $\alpha = \beta$.

Example:- 1. $\# \emptyset = 0$.

2. $\# \{0, 1, 2, \ldots, 10\} = 11$.

3. $\# \{1, 2, \ldots, n-1\} = n-1$.

4. $\# \{0, 1, 2, \dots, n-1\} = n$.

5. $\# A_k = \{1, 2, \dots, k\} = k$.

$A_6 = 6$

Ex. :- let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$. If $f: A \to B$ is defined as $f(a) = 3$, $f(b) = 2$, $f(c) = 1$. Show that $A \sim B$ (A equivalent to B).

Proof:- Now, we prove that f is bijective function

① $\forall a, b \in A$, $a \neq b \Rightarrow f(a) \neq f(b) \Rightarrow 3 \neq 2$.

$\therefore f$ is one to one.

② $\operatorname{Im} f = \{3, 2, 1\} = B$

$\therefore f$ is onto.

$\therefore f$ is bijective function

$\therefore A \sim B$.

if $\# A = \# B$ then

Lemma*:- If $A \equiv B$, then $\# A = \# B$.

Proof:- Assume that $\# A = n$ and $\# B = m$

if $A \equiv B$, then $n = m \Rightarrow \# A = \# B$.

Ex. + Let $\mathbb{N} = \{1, 2, 3, \cdots\}$

$E = \{2, 4, 6, \cdots\}$

$O = \{1, 3, 5, \cdots\}$

Show that ① $\mathbb{N} \equiv E$ ② $\mathbb{N} \equiv O$ ③ $E \equiv O$.

Solution:- ① let $f : \mathbb{N} \to E$, be defined

$f(n) = 2n, \ \forall n \in \mathbb{N}$

I- to prove f is one to one

let $n_1, n_2 \in \mathbb{N}$ s.t. $f(n_1) = f(n_2)$

$\Rightarrow 2n_1 = 2n_2$ $\div 2$

$\Rightarrow n_1 = n_2$ $\therefore f$ is one to one.

II- $\forall x \in E$, $\exists n \in \mathbb{N}$ s.t. $f(n) = x = 2n$.

$\therefore f$ is onto.

$\therefore f$ is bijective function. $\Rightarrow \mathbb{N} \equiv E$.

② let $f : \mathbb{N} \to O$, be defined $f(n) = 2n+1$ $\forall n \in \mathbb{N}$

I- to prove f is one to one

let $n_1, n_2 \in \mathbb{N}$, s.t. $f(n_1) = f(n_2)$

$\Rightarrow 2n_1 + 1 = 2n_2 + 1$

$\Rightarrow n_1 = n_2$

$\therefore f$ is one to one.

II- $\forall y \in O$, $\exists n \in \mathbb{N}$ s.t. $f(n) = y = 2n+1$

$\therefore f$ is onto $\Rightarrow f$ is bijective $\therefore \mathbb{N} \equiv O$.

③ $\mathbb{N} \equiv O \Rightarrow O \equiv \mathbb{N}$

$\mathbb{N} \equiv E \Rightarrow E \equiv \mathbb{N}$ $\Big\}$ Note (2)

$O \equiv \mathbb{N}$ and $\mathbb{N} \equiv E \Rightarrow O \equiv E$ (Note (3)).

نرمز للعدد الأساسي لـ $\mathbb{N}$ بالرمز $\#(\mathbb{N}) = \aleph_0$

Example:- Show that $\#\mathbb{N} = \#E = \#O$

Solu. Since $\#\mathbb{N} = \aleph_0$ and $\mathbb{N} \sim E$, then

$\#E = \aleph_0$ and $\mathbb{N} \sim O$, then

$\#O = \aleph_0$ (by Lemma $*$).

$\therefore \#E = \#O = \#\mathbb{N} = \aleph_0$

Ex.:- ① Show that $\#\overline{\mathbb{N}} = \aleph_0$

$\overline{\mathbb{N}} = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$

Solu.:- let $f : \mathbb{N} \to \overline{\mathbb{N}}$, $f(n) = n - 1$

اولاً نبرهن ان f دالة متباينة وشاملة حتى تكون متكافئة

I- $\forall n_1, n_2 \in \mathbb{N}$, $f(n_1) = n_1 - 1$, $f(n_2) = n_2 - 1$

$f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1 \Rightarrow n_1 = n_2$

$\therefore f$ is 1-1.

II- $\forall x \in \overline{\mathbb{N}}$, $\exists n \in \mathbb{N}$ s.t. $f(n) = x$, $\operatorname{Im} f = \overline{\mathbb{N}}$

$\therefore f$ is onto. $f(n) = n - 1 \Rightarrow x = n - 1 \Rightarrow n = x + 1 \Rightarrow f(n) = x + 1 - 1$

نعوض في الدالة $f(n) = x$

$\therefore \mathbb{N} \equiv \overline{\mathbb{N}}$ and $\#\overline{\mathbb{N}} = \#\mathbb{N} = \aleph_0$.