التحليل الرياضي (2)

Mathematical Analysis (II)

The Syllabus:

(1) Differentiation

(2) Riemann Integration

(3) Measure Theory

(4) Lebesgue Integral

References:

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(2) Rudin, W., “Principles of Mathematical Analysis”, 3rd edition, 1976, McGraw-Hill, Inc., USA.

(3) Royden. H.L., “Real Analysis”, 3rd, 1988, London.

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(5) Fusco, N., Marcellini, P., & Sbordone, C., “Mathematical Analysis: Functions of Several Real Variables and Applications”, Switzerland: Springer International Publishing, 2023.

(6) عادل غسان نعوم "مقدمة في التحليل الرياضي"، جامعة بغداد– العراق 1986 الطبعة الأولى.

(7) أنور بدرانة وآخرون، "مقدمة في التحليل الحقيقي"، دار الأمل للنشر والتوزيع، إربد، الأردن 1992.

Chapter One

الاشـتـقـاق

Differentiation

Definition (1.1): Let f be defined on an open interval I, and let $x_0 \in I$. The derivative of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided either that this limit exists or is infinite. In this case, we say that f is differentiable at x_0. The operation of obtaining $f'(x)$ from $f(x)$ is called **differentiation**.

Example (1.1): Find the derivative of the constant function $f(x) = k, \forall\, x_0 \in R$.

Solution:

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{k - k}{x - x_0} \\ &= \lim_{x \to x_0} \frac{0}{x - x_0} \\ &= \lim_{x \to x_0} 0 = 0 \end{aligned}$$

Example (1.2): Find the derivative of the linear function $f(x) = ax + b, \forall\, x_0 \in R$.

Solution:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{ax + b - ax_0 - b}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{a(x - x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} a = a$$

Example (1.3): Find the derivative of the function $f(x) = x^n$, $\forall\, x_0 \in R$.

Solution:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{x^n - {x_0}^n}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(x - x_0)(x^{n-1} + x_0 x^{n-2} + {x_0}^2 x^{n-3} + \cdots + {x_0}^{n-2} x + {x_0}^{n-1})}{x - x_0}$$

$$= \lim_{x \to x_0} (x^{n-1} + x_0 x^{n-2} + {x_0}^2 x^{n-3} + \cdots + {x_0}^{n-2} x + {x_0}^{n-1})$$

$$= {x_0}^{n-1} + x_0 {x_0}^{n-2} + {x_0}^2 {x_0}^{n-3} + \cdots + {x_0}^{n-2} x_0 + {x_0}^{n-1}$$

$$= {x_0}^{n-1} + {x_0}^{n-1} + {x_0}^{n-1} + \cdots + {x_0}^{n-1} + {x_0}^{n-1} = n\, {x_0}^{n-1}$$

Example (1.4): Find the derivative of the function $f(x) = x^2$ at $x_0 = 2$.

Solution:

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^2 - 2^2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}$$

$$= \lim_{x \to 2} (x + 2) = 4$$

Example (1.5): Find the derivative of the function $f(x) = \sqrt{x}$, $\forall\, x_0 \in (0, \infty)$

Solution:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}$$

$$= \lim_{x \to x_0} \frac{1}{(\sqrt{x} + \sqrt{x_0})} = \frac{1}{2\sqrt{x_0}}$$

Example (1.6): If it is possible, find the derivative of the function $f(x) = |x|$, $\forall\, x_0 \in R$.

Solution:

We have

$$f(x) = |x| = \begin{cases} x & , \ x > 0 \\ 0 & , \ x = 0 \\ -x & , \ x < 0 \end{cases}$$

When $x > 0$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{|x| - |x_0|}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

When $x < 0$