② Show that $\mathbb{Z} \equiv \mathbb{N}$ and $\# \mathbb{Z} = \aleph_0$

يجب ان نبرهن ان $\mathbb{Z} \equiv \overline{\mathbb{N}}$, $\overline{\mathbb{N}} \equiv \mathbb{N}$

let $f : \mathbb{Z} \longrightarrow \overline{\mathbb{N}}$, $f(n) = \begin{cases} 2n-1 & n > 0 \\ -2n & n \leq 0 \end{cases}$

$\forall n_1, n_2 \in \mathbb{Z}$, $n_1 > 0$, $n_2 > 0$, $f(n_1) = f(n_2)$

$\Rightarrow 2n_1 - 1 = 2n_2 - 1$

$\Rightarrow n_1 = n_2$

$\Rightarrow f$ is one to one.

if $n_1 \leq 0$, $n_2 \leq 0$, $f(n_1) = f(n_2)$

$\Rightarrow -2n_1 = -2n_2$

$\Rightarrow n_1 = n_2$

$\Rightarrow f$ is one to one.

let $y \in \overline{\mathbb{N}}$, $\exists n \in \mathbb{Z}$ s.t $f(n) = y$

a) $f(n) = 2n - 1 \Rightarrow y = 2n - 1 \Rightarrow 2n = y + 1 \Rightarrow n = \frac{y+1}{2}$

$\therefore f(n) = 2\left(\frac{y+1}{2}\right) - 1 = y$ is onto.

b) $f(n) = -2n \Rightarrow n = \frac{-y}{2} \Rightarrow f(n) = -2\left(\frac{-y}{2}\right) = y$ onto.

$\therefore f$ is bijective.

$\therefore \mathbb{Z} \equiv \overline{\mathbb{N}}$

$\therefore \mathbb{Z} \equiv \overline{\mathbb{N}}$, from (1) $\overline{\mathbb{N}} \equiv \mathbb{N}$.

$\therefore \mathbb{Z} \equiv \mathbb{N}$ ← هذا صعب توجد التكافؤ له نأخذ

مجموعة نسميها $\overline{N}$

$\overline{N} = N \cup \{0\}$ $\Rightarrow \aleph_0 = \# \mathbb{N} = \# \mathbb{Z}$.

Note :- 1. $Q \equiv N$, $\#(Q) = \aleph_0$.

2. $\forall k \in N$, then $N_k = \{1, 2, 3, \dots, k\}$

$\# N_1 = \{1\} = 1$, $\# N_2 = \{1, 2\} = 2$

3. $N \not\equiv N_k$.

4. A is finite set if $A = \varphi$ or $A \equiv N_n$

Ex :- $A = \{a, b, c\}$

solu. $A \neq \varphi$, $A \equiv N_3 \Rightarrow A$ is finite

5. let $A \equiv B$, then

a) A is finite $\Leftrightarrow$ B is finite

b) A is infinite $\Leftrightarrow$ B is infinite.

6. let $A = [a, b]$ and $B = [c, d]$

then

$$f(x) = \left(\frac{d - c}{b - a}\right) x + \left(\frac{bc - da}{b - a}\right)$$

قانون مهم

يستخدم عندما يعطى في السؤال فترة

let $A = [1, 2]$ and $B = [3, 6]$, then $f(x) =$ ——

Ex.:- let $A=[0,1]$, $B=[2,5]$, prove that (5)

$$\#(A)=\#(B).$$

∵ كل من A و B عبارة عن فترة

∴ نرجع على قانون العام لإيجاد الدالة $f(x)$

$$A=[\overset{a}{4},\overset{b}{7}],\quad B=[\overset{c}{-1},\overset{d}{2}]$$

$$f(x)=\frac{2-(-1)}{7-4}x+\frac{7(-1)-(2)4}{7-4}$$

$$=\frac{2+1}{3}x+\frac{-7-8}{3}$$

$$=x-5$$

Solve:- To prove $A\sim B$

$$f(x)=\left(\frac{5-2}{1-0}\right)x+\left(\frac{2-(5)(0)}{1-0}\right)$$

$$\Rightarrow f(x)=3x+2$$

I- to prove f is one to one

let $x_1, x_2 \in [0,1]$ s.t. $f(x_1)=f(x_2)$

$$\Rightarrow 3x_1+2=3x_2+2$$

$$\Rightarrow 3x_1=3x_2$$

$$\Rightarrow x_1=x_2 \Rightarrow f \text{ is one to one.}$$

$f(x)=3x+2=y$, $x\in[0,1]$, $y\in[2,5]$

$$\therefore x=\frac{y-2}{3}$$

$$2\le y\le 5$$

$$\Rightarrow 2-2\le y-2\le 5-2$$

$$\Rightarrow \frac{0}{3}\le \frac{y-2}{3}\le \frac{3}{3}$$

$$\Rightarrow 0\le \frac{y-2}{3}\le 1$$

$$\Rightarrow 0\le x\le 1$$

II- ∵ $\forall y\in[2,5]$, $\exists x\in[0,1]$ s.t. $x=\frac{y-2}{3}$

∴ f is onto $\Rightarrow$ f is bijective.

$$\therefore A\equiv B \Rightarrow \#[0,1]=\#[2,5]$$

EX. 1- let $f: (-1,1) \to (a,b)$ prove that $(-1,1) \sim (a,b)$

or $\#(-1,1) = \#(a,b)$

solve:- $f(x) = \left(\frac{b-a}{2}\right)x + \left(\frac{a+b}{2}\right)$

I- to prove f is bijective $\therefore$ to prove f is 1-1 and onto

let $x_1, x_2 \in (-1,1)$ s.t. $f(x_1) = f(x_2)$

$$\left[\left(\frac{b-a}{2}\right)x_1 + \left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)x_2 + \left(\frac{a+b}{2}\right)\right] \times 2$$

$$\Rightarrow (b-a)x_1 + \cancel{(a+b)} = (b-a)x_2 + \cancel{(a+b)}$$

$$\Rightarrow (b-a)x_1 = (b-a)x_2 \Rightarrow x_1 = x_2 \Rightarrow f \text{ is } 1-1.$$

to prove f is onto

let $y = f(x) = \left(\frac{b-a}{2}\right)x + \left(\frac{a+b}{2}\right)$

$$\Rightarrow 2y = (b-a)x + (a+b)$$

$$\Rightarrow 2y - (a+b) = (b-a)x \to x = \frac{2y-(a+b)}{(b-a)}$$

$$a < y < b \Rightarrow 2a < 2y < 2b$$

$$\Rightarrow 2a - (a+b) < 2y - (a+b) < 2b - (a+b)$$

$$\Rightarrow a - b < 2y - (a+b) < b - a$$

$$\Rightarrow -(b-a) < 2y - (a+b) < b - a$$ بالقسمة على $(b-a)$

$$-1 < \frac{2y-(a+b)}{b-a} < 1 \Rightarrow -1 < x < 1$$

II- $\forall y \in (a,b)$, $\exists x \in (-1,1)$ s.t. $y = f(x)$

$\therefore$ f is onto $\Rightarrow$ f is bijective.

$$\Rightarrow (-1,1) \sim (a,b) \Rightarrow \#(-1,1) = \#(a,b).$$

Ex.:- (1) let $A=\{a,b,c\}$

$P(A)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},A\}$

$\#(A)=3$

$\#P(A)=8$

* $\therefore$ if $\#A=n \Rightarrow \#P(A)=2^n$

(2) let $A=\emptyset$

$P(A)=\{\emptyset\}$

$\#(A)=0$

$\#P(A)=2^0=1$

If $A=\emptyset$, then $\#P(A)=$ ——

$P(A)=\{\emptyset\}$

$\#(A)=0$

$\#P(A)=2^0=1$

من خلال الامثلة السابقة لاحظنا هنالك عدد اساس غير منتهي واحد فقط هو $\aleph_0$ توجد اعداد اساس غير منتهية اخرى تختلف عن $\aleph_0$ اي هل توجد مجموعات اخرى لا تكافئ N.