Definition (1.9): (Real Numbers) **الاعـداد الـحـقـيـقـيـة**

The **real numbers** R is a complete ordered field.

Theorem (1.1): For each positive integer n and each positive number a, the equation $x^n = a$ has a unique positive solution.

Theorem (1.2): The equation $x^2 = 2$ has no solution in Q.

Proof:

Let $y \in Q \Rightarrow y = \frac{a}{b}, \ a, b \in Z, b \neq 0$

Let $y^2 = 2 \Rightarrow \frac{a^2}{b^2} = 2$

$\Rightarrow a^2 = 2b^2$ (since $b \neq 0 \Rightarrow a \neq 0$)

We have three cases:

Case (1): a, b are odd numbers

$\Rightarrow a^2, b^2$ are odd

but $a^2 = 2b^2$ and $2b^2$ is even $\Rightarrow a$ is even $\Rightarrow$ C!

Case (2): a is odd and b is even, say $b = 2d$

$\Rightarrow a^2 = 2b^2 \Rightarrow a^2 = 8d^2$

Since $8d^2$ is even $\Rightarrow a$ is even $\Rightarrow$ C!

Case (3): a is even and b is odd, say $a = 2c$

Since $a^2 = 2b^2 \Rightarrow 4c^2 = 2b^2$

$\Rightarrow b^2 = 2c^2 \Rightarrow b$ is even $\Rightarrow$ C!

$\therefore$ There is no rational number satisfy $x^2 = 2$.

Theorem (1.3): The rational numbers Q is not complete.

Proof:

Let $S = \{x \in Q : x^2 < 2\}$

$\because 1 \in S \Rightarrow S \neq \emptyset$

$\because x < 2, \forall x \in S \Rightarrow S$ is bounded above

$\because\ x^2 = 2$ has no root in Q

$\Rightarrow\ S$ has no l.u.b , thus Q is not complete.

Theorem (1.4): (**Archimedean Property**) **خاصية أرخميدس**

For every $a, b \in R$, $a > 0$, $\exists\ n \in N$ such that $na > b$.

Proof:

Let $X = \{ka : k \in N\} \subset R$, $X \neq \emptyset$

Suppose that the statement is not true

i.e. $\exists\ a,b \in R$, s.t. $\forall\ n \in N,\ na < b$

$\Rightarrow\ b$ is upper bound of X

$\Rightarrow\ X$ has l.u.b (by Completeness axiom)

Let $y = l.u.b(X)$

Since $a > 0\ \Rightarrow\ y - a < y$

$\Rightarrow\ y - a$ is not upper bound for X

$\Rightarrow\ \exists\ ma \in X$ s.t. $y - a < ma$

$\Rightarrow\ y < ma + a$

$\Rightarrow\ y < a(m+1)$

But $a(m+1) \in X\ \ \Rightarrow$ C!

$\therefore$ The statement is true.

Corollary (1.1): Let $\varepsilon > 0$, then $\exists\ n \in N$ s.t. $\frac{1}{n} < \varepsilon$.

Proof:

By using Archimedean property

Let $a = \varepsilon$ and $b = 1$

$\Rightarrow\ \exists\ n \in N$ s.t. $n\varepsilon > 1$

$\Rightarrow\ \frac{1}{n} < \varepsilon$

Corollary (1.2): Every interval in R of the form $(0,\varepsilon)$ contains infinitely many rational numbers.

Proof:

Let $\varepsilon > 0 \Rightarrow \exists\, n \in N$ s.t. $\frac{1}{n} < \varepsilon$ (by Corollary (1.1))

Since $0 < \frac{1}{n} \in Q$

$\therefore$ $(0,\varepsilon)$ has infinitely many rational numbers

---

Theorem (1.5): The field R contains a subfield isomorphic to the field Q .

Theorem (1.6): (Density of Rational Numbers Q) **كثافة الاعداد النسبية**

Every interval in R contains infinitely many of rational numbers.

Proof:

First, we prove that $\forall\, a,b \in R$, $a < b$, $\exists\, r \in Q$ s.t. $a < r < b$

Case (1): $0 < a < b$ and $b - a > 1$

Let $S = \{n \in N : n > a\} \neq \emptyset$

$\exists\, r$ the smallest number in S (Since N is well ordered)

$\Rightarrow r > a$

We have $r - 1 < r$

If $r - 1 > a \Rightarrow r - 1 \in S$

But r is smallest number in $S \Rightarrow$ C!

Thus $r - 1 \leq a$

$\Rightarrow r \leq a + 1$

Since $b - a > 1 \Rightarrow b > a + 1$

$\Rightarrow r \leq 1 + a < b$

$\Rightarrow r < b$

Thus $a < r < b$

Since $r \in N \Rightarrow r \in Q$ and we done

Case (2): $a < 0 < b$

Since $0 \in Q$ then we done

Case (3): $a < b < 0$

$\Rightarrow \quad 0 < -b < -a$

By case (1)

$\exists\, r \in Q$, s.t. $-b < r < -a \;\Rightarrow\; a < -r < b$

Second, to prove $\exists$ infinitely rational numbers between a and b

By first step,

$\exists\, r_1 \in Q$, $s.t.$ $a < r_1 < b$

Again,

$\exists\, r_2 \in Q$, $s.t.$ $a < r_2 < r_1$ and

$\exists\, r_3 \in Q$, $s.t.$ $a < r_3 < r_2$ and … so on

$\therefore$ $\exists$ infinitely rational numbers between a,b, $\forall\, a,b \in R$.

Theorem (1.7): (Density of Irrational Numbers Q') **كثافة الاعداد غير النسبية**

Every interval in R contains infinitely many irrational numbers.

Proof:

Let $I = [a,b] \subset R$

By the previous theorem,

Let $r \in Q$ s.t. $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$

$\Rightarrow \quad a < r\sqrt{2} < b \quad \Rightarrow \quad r\sqrt{2} \in I$

Clearly, $r\sqrt{2} \in Q'$ then we done

Now, to prove these numbers is infinite

Let $s_1 \in Q'$, s.t. $a < s_1 < b$

By the first step,

$\exists\, s_2 \in Q'$, s.t. $a < s_2 < s_1$ and

$\exists\, s_3 \in Q'$, s.t. $a < s_3 < s_2$ and ... so on.

$\therefore$ $\exists$ infinitely rational numbers between a,b, $\forall\, a,b \in R$.

Definition (1.10): (Dense Set) **المجموعة الكثيفة**

A subset $S \subseteq R$ is **dense** in R if $S \cap I \neq \emptyset$, $\forall$ interval $I \subset R$.

Example (1.5): The rational numbers Q is dense in real numbers R .
