$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{|x| - |x_0|}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{-x + x_0}{x - x_0} = -1$$

When $x = 0$

$$f'(x_0) = \lim_{x\to 0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x\to 0^+} \frac{x - f(0)}{x - 0}$$

$$= \lim_{x\to 0^+} \frac{x - 0}{x - 0} = 1$$

$$f'(x_0) = \lim_{x\to 0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x\to 0^-} \frac{-x - f(0)}{x - 0}$$

$$= \lim_{x\to 0^-} \frac{-x - 0}{x - 0} = -1$$

$\therefore$ f is <u>not</u> differentiable.

Theorem (1.1): If f is differentiable at x_0, then f is continuous at x_0.

Proof:

We need to prove $\lim_{x\to x_0} f(x) = f(x_0)$

Since f is differentiable

$\Rightarrow$ $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = k$ (since exists and finite)

Now,

$$\lim_{x\to x_0} f(x) - f(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}(x - x_0)$$

$$= k.0 = 0$$

$$\lim_{x\to x_0} f(x) - f(x_0) = 0$$

$$\Rightarrow \lim_{x\to x_0} f(x) = f(x_0)$$

$\therefore$ f is continuous at x_0.

Corollary (1.1): If f is discontinuous at x_0, then f is not differentiable at x_0.

Theorem (1.2): (Combination Rules)

Let f and g be defined on an interval I, and $x_0 \in I$. Then, if f and g are differentiable at x_0, so are

(i) Multiple Rule cf, for $c \in R$, and $(cf)'(x_0) = c.f'(x_0)$;

(ii) Addition Rule $f \pm g$, and $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$;

(iii) Product Rule fg, and $(fg)'(x_0) = f(x_0).g'(x_0) + f'(x_0).g(x_0)$;

(iv) Quotient Rule $\frac{f}{g}$, provided that $g(x_0) \neq 0$ and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0)f'(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}.$$

Proof:

(i) Since we have $(cf)(x) = c.f(x), \forall\, x \in D_f$; hence

$$(cf)'(x_0) = \lim_{x\to x_0} \frac{(cf)(x) - (cf)(x_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{c\big(f(x) - f(x_0)\big)}{x - x_0}$$

$$= c.\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = c.f'(x_0)$$

(ii) Since we have $(f \pm g)(x) = f(x) \pm g(x), \forall\, x \in D_f$; hence

$$(f \pm g)'(x_0) = \lim_{x \to x_0} \frac{(f \pm g)(x) - (f \pm g)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) \pm g(x) - (f(x_0) \pm g(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \pm \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) \pm g'(x_0)$$

(iii) By using definition (1.1), we have

$$(fg)'(x_0) = \lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0) + f(x)g(x_0) - f(x)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)(g(x) - g(x_0))}{x - x_0} + \lim_{x \to x_0} \frac{g(x_0)(f(x) - f(x_0))}{x - x_0}$$

$$= f(x_0).\, g'(x_0) + g(x_0).\, f'(x_0)$$

(iv) By using definition (1.1), we have

$$\left(\frac{f}{g}\right)'(x_0) = \lim_{x \to x_0} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{[f(x)g(x_0) - f(x_0)g(x) + f(x_0)g(x_0) - f(x_0)g(x_0)]/g(x)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\left[g(x_0)\big(f(x) - f(x_0)\big) - f(x_0)\big(g(x) - g(x_0)\big)\right]/g(x)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{g(x_0)}{g(x)g(x_0)} . \frac{f(x) - f(x_0)}{x - x_0} - \lim_{x \to x_0} \frac{f(x_0)}{g(x)g(x_0)} . \frac{g(x) - g(x_0)}{x - x_0})$$

$$= \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{\left(g(x_0)\right)^2}$$

Theorem (1.3): (Composition Rule)

Suppose f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then the composite function $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)).f'(x_0)$$

Proof:

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} . \frac{f(x) - f(x_0)}{f(x) - f(x_0)}$$

$$= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} . \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= g'(f(x_0)).f'(x_0)$$

Exercises (1.1) (Homework)

(1) For each of the following functions defined on R, give the set of points at which it is not differentiable

(a) $e^{|x|}$

(b) $sin|x|$

(c) $|sin\ x|$

(d) $|x| + |x - 1|$

(e) $|x^2 - 1|$

(f) $|x^3 - 8|$