Ex.:- Prove that $(-1,0) \equiv \mathbb{R}^-$

Proof:- let $f:(-1,0) \to \mathbb{R}^-$ s.t. $f(x) = \dfrac{x}{1+x}$, $\forall x \in (-1,0)$

To prove f is bijective (1-1 and onto)

I- let $x_1, x_2 \in (-1,0)$ s.t. $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$$

$$\Rightarrow x_1(1+x_2) = x_2(1+x_1)$$

$$\Rightarrow x_1 + x_1x_2 = x_2 + x_1x_2$$

$$\Rightarrow x_1 = x_2$$

$\Rightarrow f$ is one to one.

II- f is onto, since

$\forall y \in \mathbb{R}^-, \exists x \in (-1,0)$ s.t. $f(x) = y$

$$f(x) = \frac{x}{1+x} \Rightarrow y = \frac{x}{1+x} \Rightarrow x = y + xy \Rightarrow y = x - xy$$

$$\Rightarrow y = x(1-y)$$

$$\Rightarrow x = \frac{y}{1-y}$$

$$f(x) = \frac{x}{1+x} = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = \frac{\frac{y}{\cancel{1-y}}}{\frac{1-\cancel{y}+\cancel{y}}{\cancel{1-y}}} = y$$

$\therefore f$ is onto

$\Rightarrow f$ is bijective

$\therefore (-1,0) \equiv \mathbb{R}^-$

Ex.:- Prove that $(0,1) \equiv R^+$

Proof:- let $f:(0,1) \longrightarrow R^+$ s.t. $f(x) = \dfrac{x}{1-x}$, $\forall x \in (0,1)$

To prove f is bijective (1-1 and onto)

I- let $x_1, x_2 \in (0,1)$ s.t. $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$$

$$\Rightarrow x_1(1-x_2) = x_2(1-x_1)$$

$$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2$$ بإضافة x_1x_2 للطرفين

$$\Rightarrow x_1 = x_2$$

II- f is onto, since

$\forall y \in R^+$, $\exists x \in (0,1)$ s.t. $f(x) = y$

$$f(x) = \frac{x}{1-x} \Rightarrow y = \frac{x}{1-x} \Rightarrow x = y - xy \Rightarrow x + xy = y$$

$$\Rightarrow x(1+y) = y$$

$$\Rightarrow x = \frac{y}{1+y}$$

نعوض في الدالة

$$f(x) = \frac{x}{1-x} = \frac{\frac{y}{1+y}}{1-\frac{y}{1+y}} = \frac{\frac{y}{\cancel{1+y}}}{\frac{1+\cancel{y}-\cancel{y}}{\cancel{1+y}}} = y$$

$\therefore f$ is onto

$\therefore f$ is bijective

$\Rightarrow (0,1) \equiv R^+$.

Ex.:- prove that $(-1,1) \equiv R$

proof:- let $f:(-1,1) \longrightarrow R$

H.W
$$f(x) = \begin{cases} \dfrac{x}{1+x} & -1 < x \leq 0 \\ \dfrac{x}{1-x} & 0 < x < 1 \end{cases}$$

Ex.:- prove that $\mathbb{R} \equiv (a,b)$

proof:- from Ex.: $(-1,1) \equiv R$

and from Ex.: $(-1,1) \equiv (a,b)$

بما ان $\equiv$ علاقة تكافؤ تكون متعدية اي ان

$$R \equiv (-1,1) \wedge (-1,1) \equiv (a,b)$$

$$\Rightarrow R \equiv (a,b)$$

$$\therefore \#R = \#(a,b)$$

نرمز للعدد الاساسي لاي فترة مفتوحة ومغلقة هو c

$$\therefore \#\mathbb{R} = c = \#(a,b)$$

سؤال :- الى حد الآن حصلنا على العدد الاساسي $\aleph_0$, c غير منتهيين، هل توجد اعداد اساسية اخرى غير منتهية تختلف عن $\aleph_0$, c لقد اجاب عن هذا السؤال كانتور.

Cantor's theorem

State Cantor's theorem —— (8)

let A be a set, then $A \nsim P(A)$ <u>or</u> $\#(A) \neq \#P(A)$.

Proof:- 1. let $A = \emptyset \Rightarrow P(A) = \{\emptyset\}$.

$$\#A = 0 \Rightarrow \#P(A) = 2^0 = 1.$$

$$\therefore \#A \neq \#P(A) \qquad \therefore A \nsim P(A).$$

2. let $A \neq \emptyset$ and suppose that $A \sim P(A)$, then $\exists$ a function $f : A \to P(A)$, $f(a) \in P(A)$, $\forall a \in A$

$$\therefore f(a) \subseteq A.$$

$\therefore$ a is good element if $a \in f(a)$.

a is not good element if $a \notin f(a)$.

assume that $B = \{a \in A : a \notin f(a)\}$

$$\therefore B \subseteq A \text{ and } B \in P(A).$$

$\because f$ is onto then $\exists\, b \in A$ s.t. $f(b) = B$

I. If b is good element

$\therefore b \in f(b) = B \Rightarrow b$ is not good element C!

II. If b is not good element

$\therefore b \notin f(b) = B \Rightarrow b$ is good element C!

$$\therefore A \nsim P(A)$$

<u>or</u>

$$\#A \neq \#P(A).$$

Def.:- 1. Let A and B are sets, then

$\#(A) \leq \#(B)$, if $\exists f: A \to B$ is one to one mapping.

2. $\#(A) < \#(B)$ if $\exists f: A \to B$ is onto

ولا يوجد دالة متباينة وشاملة في نفس الوقت اي ان

دالة متباينة

$\#(A) < \#(B)$ iff (1) $\#(A) \leq \#(B)$.

(2) $\#(A) \neq \#(B)$. لا توجد دالة 1-1 وشاملة في آن واحد

Ex.(1):- Let $N_k = \{1, 2, 3, \ldots, k\}$, $\#N_k = K$, $\#N = \aleph_0$

prove that $K < \aleph_0$

Proof:- let $f: N_k \to N$

بما ان $N_k \subset N$ فان f تسمى دالة الاحتواء (inclusion mapping) وتكون هذه الدالة دائماً متباينة

$\Rightarrow \#N_k \leq \#N \Rightarrow K \leq \aleph_0$

since $N_k \neq N \Rightarrow \#N_k \neq \#N \Rightarrow K \neq \aleph_0 \Rightarrow K < \aleph_0$

Ex(2):- Show that $\aleph_0 < c$

proof:- let $J = (0,1)$, $\#(J) = c$, $\#N = \aleph_0$

and $f: N \to J$, such that

$$f(n) = \frac{1}{n+1}, \quad \forall n \in N$$