Chapter Two
متتابعات الاعداد الحقيقية
Sequences of Real Numbers

Definition (2.1): (Sequence) المتتابعة

Let $S \neq \emptyset$, every function $f_n : N \to S$ is called a sequence in S and we write it $< a_n >$ or $\{a_n\}_n^\infty$ or (a_n) .

Example (2.1):

(1) $< \sqrt{n} > = < 1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots >$

(2) $< \frac{1}{n} > = < 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots >$

(3) $< (-1)^{n+1} \frac{1}{n} > = < 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ \ldots >$

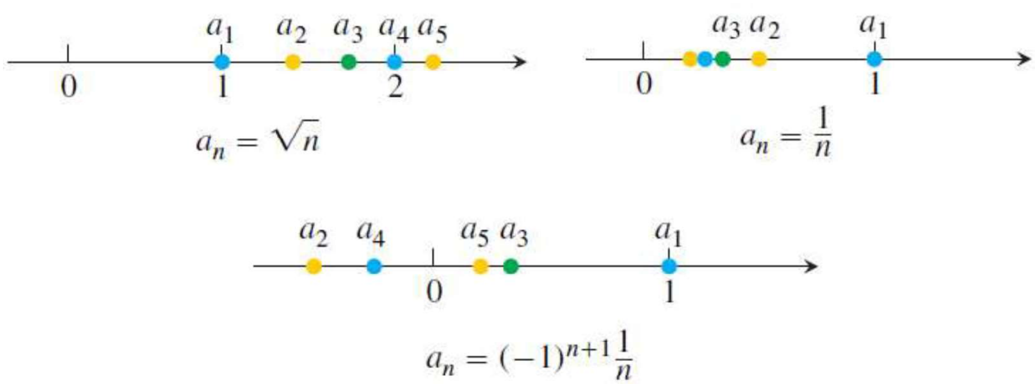

Figure (2.1). Sequences can be represented as points on the real line

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Definition (2.2): (Convergent Sequence) المتتابعة المتقاربة

The sequence $< a_n >$ **converges** to a_0 if

$\forall\ \varepsilon > 0$, $\exists$ positive integer $k = k(\varepsilon)\ s.t.\ |a_n - a_0| < \varepsilon$, $\forall\ n > k$.

i.e. $\lim_{n\to\infty} a_n = a_0$ or $a_n \to a_0$

If no such number a_0 exists, we say that $< a_n >$ diverges

Example (2.2):

(1) $< \sqrt{n} > = < 1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots >$ diverge

(2) $< \frac{1}{n} > = < 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots > \to 0$ as $n \to \infty$

(3) $< (-1)^{n+1} \frac{1}{n} > = < 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots > \to 0$ as $n \to \infty$

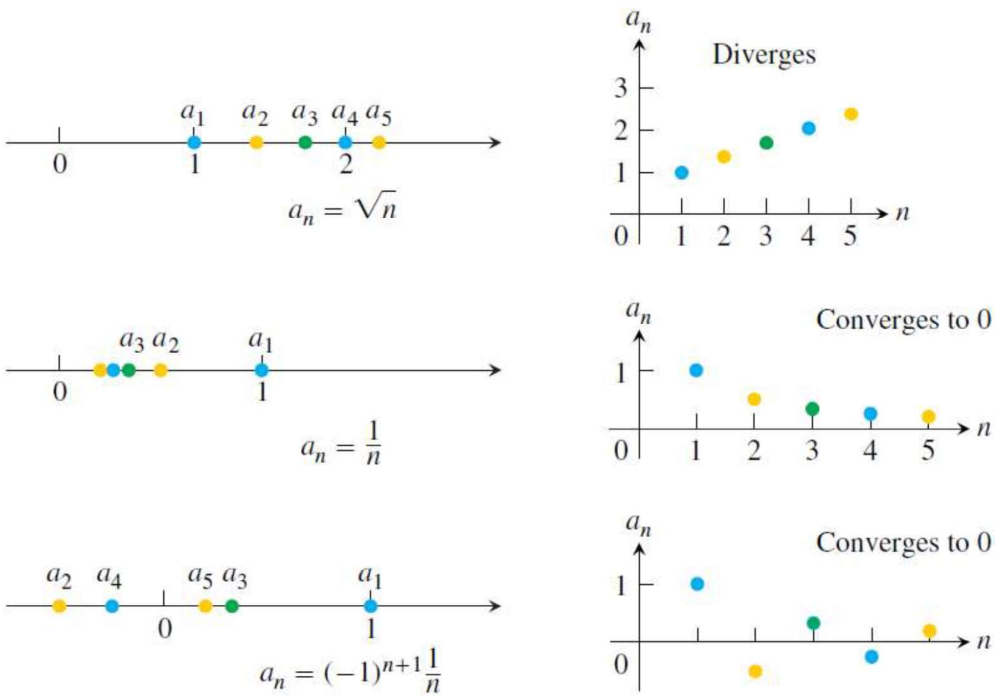

Figure (2.2). Sequences can be represented as points on the real line or as points in the plane

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Theorem (2.1): The convergent point of a sequence is unique.

Proof: Let $< a_n >$ be a sequence and

$a_n \to a_0$ and $a_n \to b_0$ s.t. $a_0 \neq b_0$

Let $|a_0 - b_0| < d$, $d > 0$

$\because\ < a_n > \to a_0 \Rightarrow \exists$ positive integer k_1, s.t. $|a_n - a_0| < \frac{d}{2}$, $\forall\, n > k_1$

Similarly,

$\because\ < a_n > \to b_0 \Rightarrow \exists$ positive integer k_2, s.t. $|a_n - b_0| < \frac{d}{2}$, $\forall\, n > k_2$

Now, let $k = max\{k_1, k_2\}$

So,

$$d = |a_0 - b_0| = |a_0 - a_k + a_k - b_0|$$
$$\leq |a_0 - a_k| + |a_k - b_0|$$
$$< \frac{d}{2} + \frac{d}{2} = d \quad \Rightarrow \quad C!$$

$\therefore \; a_0 = b_0$

$\Rightarrow$ The convergent point is unique

Definition (2.3): (Bounded Sequence) المتتابعة المقيدة

A sequence $< a_n >$ is said to be **bounded** iff $\exists\, M \in R$ s.t. $|a_n| \leq M, \forall\, n$.

Example (2.3):

(1) $< \frac{n+2}{2n-1} > = < 3, \frac{4}{3}, 1, \frac{6}{7}, \ldots >$ is bounded by 3

(2) $< (-1)^{n+1} > = < 1, -1, 1, -1, \ldots >$ is bounded by 1

(3) $< \frac{2}{n^2} > = < 2, \frac{1}{2}, \frac{2}{9}, \frac{1}{8}, \ldots >$ is bounded by 2

Definition (2.4): (Monotonic Sequence) المتتابعة الرتيبة

The sequence $< a_n >$ is said to be **monotonic sequence** if $a_n \leq a_{n+1}$, $\forall\, n$ or $a_n \geq a_{n+1}$, $\forall\, n$.

Example (2.4):

(1) $< \frac{n+1}{n^2} > = < 2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots >$ monotonic sequence (decreasing seq.) متناقصة

(2) $< \frac{2^n-1}{2^n} > = < \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots >$ monotonic sequence (increasing seq.) متزايدة

Theorem (2.2): Every bounded monotone sequence is convergent.

Proof: Let $< a_n >$ be a bounded monotone sequence

<u>Case (1)</u>:

$a_n \leq a_{n+1}, \;\; \forall\, n$ and $|a_n| < M$, $\forall\, n$

Let $S = \{a_n : n \in N\}$