(2) Use the definition of derivative to calculate the derivatives of the following functions at the indicated points

(a) $f(x) = x^3$ at $x = 2$;

(b) $g(x) = x + 2$ at $x = a$;

(c) $f(x) = x^2 \cos x$ at $x = 0$;

(d) $r(x) = \frac{3x+4}{2x-1}$ at $x = 1$.

(3)

(a) Let $h(x) = \sqrt{x}$ for $x \geq 0$. Use the definition of derivative to prove $h'(x) = \frac{1}{2\sqrt{x}}$ for $x > 0$.

(b) Let $f(x) = \sqrt[3]{x}$ for $x \in R$ and use the definition of derivative to prove $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ for $x \neq 0$.

(4) Let $f(x) = x^2 \, sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$

(a) Use the definition to show f is differentiable at $x = 0$ and $f'(0) = 0$.

(c) Show f' is not continuous at $x = 0$.

Theorem (1.4): (Local Extremum Theorem) or (Fermat's Principle)

Let $f: (a, b) \rightarrow R$ be a function and suppose that f has a local extremum (maximum or minimum) at a point $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.

Proof: Let f has local maximum at $x_0 \in (a, b)$

Then $\exists \ \varepsilon > 0$, such that

$|x - x_0| < \varepsilon \ \rightarrow \ f(x) \leq f(x_0)$, $\forall \ x \in (a, b)$

$\Rightarrow \ \ f(x) - f(x_0) \leq 0$

Since $|x - x_0| > 0$

either $x > x_0$

$\Rightarrow \quad x - x_0 > 0 \quad \Rightarrow \quad \frac{f(x)-f(x_0)}{x-x_0} \leq 0$

$\Rightarrow \quad f'(x_0) \leq 0$ ………. (1)

Or $x < x_0$

$\Rightarrow \quad x - x_0 < 0 \quad \Rightarrow \quad \frac{f(x)-f(x_0)}{x-x_0} \geq 0$

$\Rightarrow f'(x_0) \geq 0$ ………. (2)

From (1) and (2), we have

$f'(x_0) = 0$

Theorem (1.5): (Rolle's Theorem)

Suppose that $f: [a, b] \to R$ is continuous function that is also differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof: If f is constant function on $[a, b]$

$\Rightarrow \quad f'(x) = 0, \ \forall \ x \in (a, b)$;

$\Rightarrow \quad c$ may be any point in (a, b).

If f is non-constant function on $[a, b]$, then

either maximum or minimum (or both) of f on $[a, b]$ is different from the value $f(a) = f(b)$.

Since one of the extrema exists at some point $c \in (a, b)$, then by Theorem (1.4) $f'(c) = 0$.

Example (1.7): Verify that the conditions of Rolle's Theorem are satisfied by the function $f(x) = 3x^4 - 2x^3 - 2x^2 + 2x$, $x \in [-1,1]$, and determine a value of c in $(-1,1)$ for which $f'(c) = 0$.

Solution:

Since f is a polynomial function

$\Rightarrow f$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$.

Also,

$f(-1) = 3(-1)^4 - 2(-1)^3 - 2(-1)^2 + 2(-1) = 1$

$f(1) = 3(1)^4 - 2(1)^3 - 2(1)^2 + 2(1) = 1$

$\Rightarrow f(-1) = f(1) = 1$

$\therefore f$ satisfies the conditions of Rolle's Theorem on $[-1,1]$.

Now,

$f'(x) = 12x^3 - 6x^2 - 4x + 2$

$\quad = 12\left(x^2 - \frac{1}{3}\right)\left(x - \frac{1}{2}\right);$

We have the points $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{2}$ in $(-1,1)$. Any of these number will serve for c.

Theorem (1.6): (Mean Value Theorem)

Suppose that $f: [a, b] \to R$ is a continuous function that is also differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

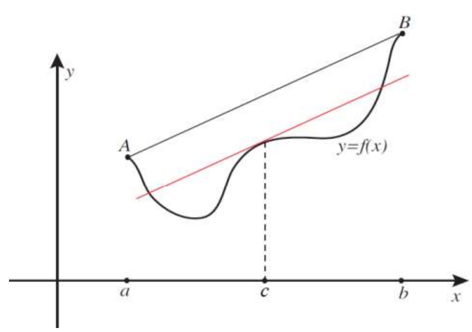

Figure (1)

Proof:

Let $L(x)$ be the linear function through $(a, f(a))$ and $(b, f(b))$,

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Consider the function $g(x) = f(x) - L(x)$.

Since f and L are continuous and differentiable on $[a, b]$

$\Rightarrow$ g is continuous and differentiable on $[a, b]$

Moreover, $g(a) = g(b) = 0$.

So by Rolle's Theorem, there is $c \in (a, b)$ such that

$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$

$\Rightarrow$ $f'(c) = \frac{f(b)-f(a)}{b-a}$

Remark (1.1): Rolle's Theorem is the special case of the Mean Value Theorem where $f(a) = f(b)$.

Example (1.8): Verify that the conditions of Mean Value Theorem are satisfied by the function $f(x) = x^2 - 6x + 4$, $x \in [-1,1]$, and find the value of c that satisfied the conclusion of the theorem.

Solution:

Since f is a polynomial function

$\Rightarrow f$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$.

$\therefore f$ satisfies the conditions of Mean Value Theorem on $[-1,1]$.

We have

$f(a) = f(-1) = (-1)^2 - 6(-1) + 4 = 11$

$f(b) = f(1) = (1)^2 - 6(1) + 4 = -1$

Now,

$f'(c) = \frac{f(b)-f(a)}{b-a}$