$\forall\, n_1, n_2 \in W \;,\; f(n_1) = f(n_2)$

$$\Rightarrow \frac{1}{n_1+1} = \frac{1}{n_2+1}$$

$$\Rightarrow n_1 = n_2 \qquad \therefore f \text{ is one to one.}$$

$\because$ f دالة متباينة فإن

$$\#(N) \leq \#(J) \Rightarrow \aleph_0 \leq C$$

$$\text{since } \mathbb{N} \not\equiv J \Rightarrow \#N \neq \#J \Rightarrow \aleph_0 \neq C$$

$$\therefore \aleph_0 < C$$

Corollary:- let A be a set $\#(A) < \#(P(A))$.

Proof:- (1) let $f: A \to P(A)$, $f(a) = \{a\}$, $\forall a \in A$

f(a) متباينة

f دالة متباينة (برهان)

$\Rightarrow \#(A) \leq \#P(A)$

$\{a\} \neq \{b\}$ لان فرضنا العناصر تكون فردة

(2) From cantor's theorem . $\Rightarrow f(a) \neq f(b)$

$$A \not\sim P(A) \Rightarrow \#(A) \neq \#P(A)$$

From ① and ② we get $\#(A) < \#P(A)$.

Theorem:- let A and B are sets , if

$$A \equiv B \text{ , then } P(A) \equiv P(B).$$

Note:- If $\#(A)=\alpha$, $\#(B)=\beta$, then

1- $\alpha \leq \alpha$

2- $\alpha \leq \beta$ and $\beta \leq \alpha \Rightarrow \alpha = \beta$.

3- $\alpha \leq \beta$ and $\beta \leq \gamma \Rightarrow \alpha \leq \gamma$.

4- $\#(A)=\alpha$, $\# P(A) = 2^{\alpha}$.

Proof(2):- $\because \alpha \leq \beta \Rightarrow \#(A) \leq \#(B) \Rightarrow f: A \to B$ is 1-1

$\because \beta \leq \alpha \Rightarrow \#(B) \leq \#(A) \Rightarrow g: B \to A$ is 1-1

$\therefore \quad A \equiv B$

$\Rightarrow \#(A) = \#(B)$

$\Rightarrow \alpha = \beta$.

Definition:- The set A is said to be (countible) if $A \sim B$ where $B \subset \mathbb{N}$ and we said A is (infinite countible) set if $A \sim \mathbb{N}$.

Definition:- Let A and B are sets , $\# A = \alpha$, $\# B = \beta$, with $A \cap B = \emptyset$, Then

$$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$$ قانون

Some properties of Cardinal numbers

let α, β, γ be cardal numbers :- هذه عبارة عن الاعداد

1- $\alpha + \beta = \beta + \alpha$.

2. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

3. $\alpha + 0 = \alpha$.

4. $c + \aleph_0 = c$ and $\alpha = \alpha + \aleph_0$.

c اكبر من $\aleph_0$; α اكبر من $\aleph_0$

Ex. :- Show that $\aleph_0 + \aleph_0 = \aleph_0$.

proof:- let $\mathbb{N} = E \cup O$, $E \cap O = \emptyset$

$$\therefore \# \mathbb{N} = \# (E \cup O)$$

$$\# \mathbb{N} = \# E + \# O - \# (E \cap O)$$

$$\aleph_0 = \aleph_0 + \aleph_0 - 0$$

$$\Rightarrow \aleph_0 = \aleph_0 + \aleph_0$$

Ex. :- Show that $k + \aleph_0 = \aleph_0$.

proof:- let $N_k = \{1, 2, 3, \dots, k\}$

$S_{k+1} = \{k+1, k+2, k+3, \dots\}$ هذه المجموعة كانت المكمل لـ N_k

Now, $N_k \cup S_{k+1} = N$, $N_k \cap S_{k+1} = \emptyset$

$$\therefore \# N = \# (N_k \cup S_{k+1})$$

$$\# N = \# N_k + \# S_{k+1} - \# (N_k \cap S_{k+1})$$

عدد العناصر من المجموعة ; لانه الى ما لا نهاية

$$\aleph_0 = k + \aleph_0 - 0$$

$$\Rightarrow \aleph_0 = k + \aleph_0 .$$

Note:- $\#(\mathbb{R}) = c$.

Example:- prove that $c + \aleph_0 = c$

Proof:- let $R_1 = \mathbb{R} - E$

i.e $\mathbb{R} = R_1 \cup E$, $R_1 \cap E = \emptyset$

we have $\#(\mathbb{R}) = c$, $\#(E) = \aleph_0$

$\#(\mathbb{R}) = \#(R_1 \cup E)$

$\#(\mathbb{R}) = \#(R_1) + \#(E) - \#(R_1 \cap E)$

$\Rightarrow c = \#(R_1) + \aleph_0 - 0$

$\Rightarrow c = \#(R_1) + \aleph_0$ --- ①

Now, $\mathbb{N} = E \cup O$

$\Rightarrow \#(\mathbb{N}) = \#(E) + \#(O) - \#(E \cap O)$

$\Rightarrow \aleph_0 = \#(E) + \aleph_0 - 0$

$\Rightarrow \aleph_0 = \#(E) + \aleph_0$ --- ②

from ① $c = \#(R_1) + \underline{\aleph_0}$ → نعوض عنها في ②

$\Rightarrow c = \#(R_1) + \#(E) + \aleph_0$

$c = \#(R_1 \cup E) + \aleph_0$

$c = \#(\mathbb{R}) + \aleph_0$

$c = c + \aleph_0$

Ex.:- prove that $c = c + k$

proof:- من الملاحظة السابقة الجزء (4) نحصل على ان

$$c = c + x_0$$
$$= c + (x_0 + k) \quad \text{من المثال (2)}$$
الخاصية التجميعية
$$= (c + x_0) + k$$
$$= c + k.$$

Ex.:- prove that $\alpha = \alpha + k$, where A is any infinite set and $\# A = \alpha$.

proof:- من الملاحظة السابقة الجزء الثاني من 4 نحصل على ان

$$\alpha = \alpha + x_0$$
$$= \alpha + (x_0 + k) \quad \text{من المثال (2)}$$
$$= (\alpha + x_0) + k$$
$$= \alpha + k$$

Ex. Find 1. $\#[a,b]$ 2. $\#(a,b]$

proof:- $[a,b] = [a] \cup (a,b) \cup [b]$

حيث ان $[a] \cap (a,b) \cap [b] = \emptyset$

$$\therefore 1.\ \#[a,b] = \#[a] + \#(a,b) + \#[b] - \#([a] \cap (a,b) \cap [b])$$
$$= 1 + c + 1 - 0 \qquad (\#(\emptyset) = 0)$$
$$= 2 + c$$
$$= c$$

$$2.\ \#(a,b] = \#((a,b) \cup [b])$$
$$= \#(a,b) + \#[b] - \#((a,b) \cap [b])$$
$$= c + 1 - 0 \qquad (\#(\emptyset) = 0)$$
$$= c + 1$$
$$= c.$$