$\because\ |a_n| < M\ ,\ \forall\, n\ \Rightarrow\ S$ is bounded above

$\Rightarrow\ S$ has l.u.b (By Completeness axiom)

Let $b = l.u.b(S)$ and for $\varepsilon > 0$ , we have $b - \frac{\varepsilon}{2} < b$

Hence $b - \frac{\varepsilon}{2}$ not upper bound

$\because\ \exists\, a_k$ s.t. $a_k > b - \frac{\varepsilon}{2}$ and $< a_n >$ is increasing then

$a_n \geq a_k\ , \forall\, n > k$

$\Rightarrow\ b - \frac{\varepsilon}{2} < a_n\ ,\ \forall\, n > k$

$\Rightarrow\ |a_n - b| < \frac{\varepsilon}{2}\ ,\ \forall\, n > k$

$\Rightarrow\ a_n \rightarrow b$

$\therefore\ < a_n >$ is convergent

<u>Case (2)</u>:

$a_n \geq a_{n+1}\ , \forall\, n$ and $|a_n| < M\ , \forall\, n$

The proof is similar to case (1).

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**Example (2.5):** Monotone sequence which is not convergent.

The sequence $< 1, 2, 3, \ldots >$ is

(1) Monotone

(2) Bounded below, not bounded above

(3) Not convergent

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**Example (2.6):** The sequence $< 1, 1.4, 1.41, \ldots >$ in $Q$ is

(1) Bounded by 2

(2) Convergent to $\sqrt{2}$

Thus $\sqrt{2} \notin Q$ then the sequence is not convergent in $Q$ .

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**Example (2.7):** Sequence bounded not monotone.

The sequence $< (-1)^n >$ is

(1) Bounded by 1

(2) Not monotone

(3) Not convergent

---

**Definition (2.5): (Cauchy Sequence)** متتابعة كوشي

A sequence $< a_n >$ of real numbers is called a Cauchy sequence if for every $\forall\, \varepsilon > 0$, $\exists$ a positive integer $k$ s.t. $|a_m - a_n| < \varepsilon$ , $\forall\, m, n > k$.

---

**Theorem (2.3):** Every convergent sequence is Cauchy sequence.

**Proof:**

Let $< a_n >$ be convergent sequence and let $a_n \to a_0$

Let $\varepsilon > 0$ since $a_n \to a_0$ then $\exists\, k \in N$ s.t. $|a_n - a_0| < \frac{\varepsilon}{2}$ ,$\forall\, n > k$

Now, $\forall\, n, m \in N$ , $n, m > k$

$$|a_n - a_m| = |a_n - a_0 + a_0 - a_m|$$
$$\leq |a_n - a_0| + |a_m - a_0|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow\ |a_n - a_m| < \varepsilon$ , $\forall\, n, m > k$

$\therefore\ < a_n >$ is Cauchy sequence

---

**Theorem (2.4):** Every Cauchy sequence is bounded.

**Proof:** Let $< a_n >$ be a Cauchy sequence

Let $\varepsilon = 1$

$\therefore\ \ \exists\, k \in N$, s.t. $|a_n - a_{n+1}| < 1$ , $\forall\, n > k$

$\therefore\ |a_n| < |a_{n+1}| + 1$ , $\forall\, n$

Let $M = max\{|a_1|, |a_2|, \dots, |a_n|, |a_{n+1}| + 1\}$

$\therefore\ |a_n| \leq M$ , $\forall\, n \in N$

$\therefore\ \ |a_n|$ is bounded

---

**Example (2.8):** Not Cauchy sequence but bounded.

The sequence $< (-1)^n >$ is bounded by 1

But $< (-1)^n >$ is not Cauchy

Let $\varepsilon = \frac{1}{2}$, let $k \in N$

$|a_n - a_{n+1}| = |(-1)^n - (-1)^{n+1}| = |2| = 2 > \frac{1}{2}$

$\therefore < (-1)^n >$ not Cauchy sequence.

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**Definition (2.6): (Nested Sequence)** المتتابعة المعشعشة

Let $< I_n >$ be a sequence of intervals $< I_n >$ is called **nested sequence** if $I_{n+1} \subseteq I_n$ , $\forall\, n$ .

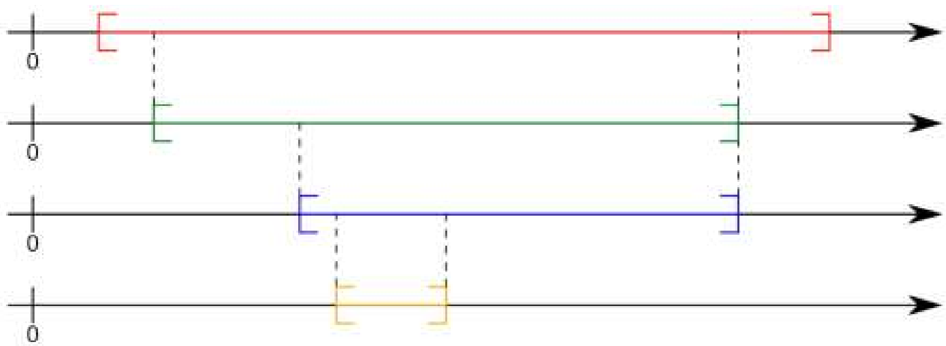


**Figure (2.3).** A sequence of nested intervals

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**Example (2.9):**

$< I_n > = < \left[0, \frac{1}{n}\right] >$ is nested sequence

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**Theorem (2.5): (Nested Intervals Theorem)** نظرية الفترات المعشعشة

Let $< I_n >$ be a nested sequence of closed intervals in $R$ such that $|I_n| \to 0$ . Then $\bigcap_n I_n \neq \emptyset$ and $\bigcap_n I_n$ contains only one element.

**Proof:**

Let $I_n = [a_n, b_n]$

Let $S_1 = \{a_1, a_2, \ldots\}$ and $S_2 = \{b_1, b_2, \ldots\}$

Now, $a_n < b_n$ , $\forall\, n$

$a_m \geq a_n$ and $b_m \leq b_n$ , $\forall\, m \geq n$

$\Rightarrow\ a_n \leq a_m < b_m \leq b_n$

$\Rightarrow\ a_m < b_n$ , $\forall\, m \geq n$

$\therefore$ every element in $S_2$ is upper bound for $S_1$

$\Rightarrow S_1$ has l.u.b (By the completeness axiom)

Let $y = l.u.b(S_1)$

Since $a_n \leq y$ , and $y \leq b_n$ , $\forall\, n$

$\Rightarrow\ a_n \leq y \leq b_n$ , $\forall\ n$

$\Rightarrow\ y \in I_n$ , $\forall\ n$

$\therefore\ y \in \bigcap_n I_n \ \Rightarrow\ \bigcap_n I_n \neq \emptyset$

Assume $y_1 , y_2 \in \bigcap_n I_n$ , $y_1 \neq y_2$

Since $|I_n| \to 0 \ \Rightarrow\ |\bigcap_n I_n| \to 0$

$\Rightarrow\ |y_1 - y_2| = 0 \ \Rightarrow\ y_1 = y_2$

$\therefore \bigcap_n I_n$ contains only one element

---

**Example (2.9):** $< I_n > = < \left[0, \frac{1}{n}\right] >$

(1) $I_{n+1} \subseteq I_n$ nested intervals

(2) $|I_n| = \left|\frac{1}{n}\right| \to 0$

(3) $\bigcap_n I_n = \{0\}$

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**Theorem (2.6):** Every Cauchy sequence in $R$ is convergent.

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**Theorem (2.7):** If $F$ is an ordered field in which every Cauchy sequence is convergent, then $F$ is complete.

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