$\Rightarrow \quad 2c - 6 = \frac{-1-11}{1-(-1)} \qquad \Rightarrow \qquad c = 0$

Corollary (1.2): Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).

Corollary (1.3): Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$, $\forall\, x \in (a, b)$.

Definition (1.2): Let f be a real-valued function defined on an interval I. We say

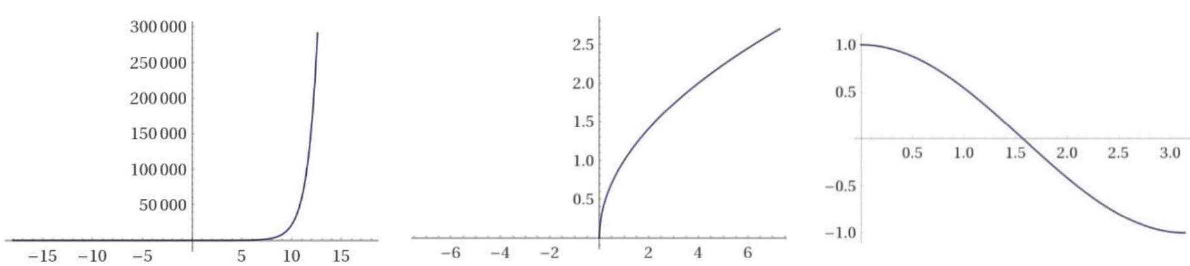
(1) f is **increasing** on I, if $x, y \in I$ and $x < y$ imply $f(x) \leq f(y)$,

(2) f is **strictly increasing** on I, if $x, y \in I$ and $x < y$ imply $f(x) < f(y)$,

(3) f is **decreasing** on I, if $x, y \in I$ and $x < y$ imply $f(x) \geq f(y)$,

(4) f is **strictly decreasing** on I, if $x, y \in I$ and $x < y$ imply $f(x) > f(y)$.

Example (1.9): The functions e^x on R and $\sqrt{x}$ on $[0, \infty)$ are strictly increasing. The function $cos(x)$ is strictly decreasing on $[0, \pi]$.



e^x $\qquad \sqrt{x}$ $\qquad cos(x)$

Figure (2)

Corollary (1.4): Let f be a differentiable function on (a, b). Then

(1) f is **increasing** if $f'(x) \geq 0$ for all $x \in (a, b)$;

(2) f is **strictly increasing** if $f'(x) > 0$ for all $x \in (a, b)$;

(3) f is **decreasing** if $f'(x) \leq 0$ for all $x \in (a, b)$;

(4) f is **strictly decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Corollary (1.5): Suppose that $f: [a, b] \to R$ is a continuous function that is differentiable on (a, b). If $f'(x) \neq 0$, for any $x \in (a, b)$, then f is injective.

Theorem (1.7): (Intermediate Value Theorem for Derivatives)

Let f be a differentiable function on (a, b). If $a < x < y < b$, and if $f'(x) < c < f'(y)$, there exists $k \in (x, y)$ such that $f'(k) = c$.

Applications of Differentiation

Theorem (1.8): (L'Hôpital's Rule)

Let f and g be differentiable on a neighborhood of the point c, at which $f(c) = g(c) = 0$ or $\pm\infty$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} \text{ exists and equals } \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that this last limit exists.

Example (1.10): Prove that $\lim_{x \to \frac{\pi}{2}} \frac{cos(3x)}{sin(x) - e^{\cos(x)}}$ exists and determine its value.

Solution:

Let $f(x) = cos(3x)$ and $g(x) = sin(x) - e^{\cos(x)}$, for $x \in R$

Then f and g are differentiable on R, and

$f\left(\frac{\pi}{2}\right) = g\left(\frac{\pi}{2}\right) = 0;$

Now,

$$\lim_{x \to \frac{\pi}{2}} \frac{f\prime(x)}{g\prime(x)} = \lim_{x \to \frac{\pi}{2}} \frac{-3\,sin(3x)}{cos(x)+sin(x).e^{cos(x)}} = \frac{3}{0+1(1)} = 3$$

It follows from L'Hôpital's Rule that the limit of $\frac{cos(3x)}{sin(x)-e^{\cos(x)}}$ exist and that its value is 3.

Example (1.11): Prove that $\lim_{x \to 0} \frac{x^2}{cosh(x)-1}$ exists and determine its value.

Solution:

Let $f(x) = x^2$ and $g(x) = cosh(x) - 1$, for $x \in R$

Then f and g are differentiable on R, and

$f(0) = g(0) = 0;$

$$\lim_{x \to 0} \frac{f\prime(x)}{g\prime(x)} = \lim_{x \to 0} \frac{2x}{sinh(x)}$$

We have $f'(0) = g'(0) = 0;$

Now,

$$\lim_{x \to 0} \frac{f\prime\prime(x)}{g\prime\prime(x)} = \lim_{x \to 0} \frac{2}{cosh(x)} = \frac{2}{1} = 2$$

It follows from L'Hôpital's Rule that the limit of $\frac{x^2}{cosh(x)-1}$ exist and that its value is 2.

Exercises (1.2) : (Homework)

(1) Determine whether the conclusion of the Mean Value Theorem holds for the following functions on the specified intervals.

(a) x^2 on $[-1, 2]$,

(b) $sin\,x$ on $[0, \pi]$,

(c) $|x|$ on $[-1, 2]$,

(d) $\frac{1}{x}$ on $[-1, 1]$,

(e) $\frac{1}{x}$ on $[1,3]$,

(2) Suppose f is differentiable on R and $f(0)=0$, $f(1)=1$ and $f(2)=1$.

(a) Show $f'(x)=\frac{1}{2}$ for some $x\in(0,2)$.

(b) Show $f'(x)=\frac{1}{7}$ for some $x\in(1,2)$.

(3) Let f be defined on R, and suppose $|f(x)-f(y)|\leq(x-y)^2$ for all $x,y\in R$. Prove f is a constant function.

(4) Show $\sin x\leq x$ for all $x\geq 0$. *Hint:* Show $f(x)=x-\sin x$ is increasing on $[0,\infty)$.

Definition (1.3): (Taylor Polynomial) متعددة حدود تايلر

Let f be n-times differentiable on an open interval containing the point a. Then the **Taylor polynomial** of degree n for f at a is the polynomial

$$T_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example (1.12): Determine the Taylor polynomials $T_1(x)$, $T_2(x)$ and $T_3(x)$ for the function $f(x)=\sin x$ at each of the following points:

(i) $a=0$;

(ii) $a=\frac{\pi}{2}$.

Solution:

$f(x)=\sin x$, $\quad f(0)=0$, $\quad f\left(\frac{\pi}{2}\right)=1$;

$f'(x)=\cos x$, $\quad f'(0)=1$, $\quad f'\left(\frac{\pi}{2}\right)=0$;

$f''(x)=-\sin x$, $\quad f''(0)=0$, $\quad f''\left(\frac{\pi}{2}\right)=-1$;

$f'''(x)=-\cos x$, $\quad f'''(0)=-1$, $\quad f'''\left(\frac{\pi}{2}\right)=0$.