Note:- $\#(A \times B) = (\# A)(\# B)$

Ex.:- prove that $\alpha = \aleph_0 + \alpha$

Q:- If $\#(A) = \alpha$ where A is any infinite set

Proof:- let A be any infinite set Then $\alpha + \aleph_0 = \alpha$

i.e. $\#(A) = \alpha$.

and let $S \subset A$ s.t. $S \equiv W$

and $S = \{x_1, x_2, x_3, x_4, \dots\}$

$S_1 = \{x_1, x_3, x_5, \dots\}$

$S_2 = \{x_2, x_4, x_6, \dots\}$

$\Rightarrow S_1 \cup S_2 = S \quad , \quad S_1 \cap S_2 = \emptyset$

and let $A_1 = A - S_1$

$\Rightarrow A_1 \cup S_1 = A$ and $S_1 \cap A_1 = \emptyset$

$\#(A) = \#(A_1 \cup S_1) = \#(A_1) + \#(S_1) - \#(A_1 \cap S_1)$

$\Rightarrow \#(A) = \#(A_1) + \#(S_1) - 0$

$\Rightarrow \alpha = \#(A_1) + \aleph_0 \text{ ---- } ①$

Now, $S = (S_1 \cup S_2) \quad , \quad S_1 \cap S_2 = \emptyset$

$\underset{\aleph_0}{\underset{\downarrow}{\#(S)}} = \#(S_1 \cup S_2) = \#(S_1) + \#(S_2) - \#(S_1 \cap S_2)$

$= \#(S_1) + \aleph_0 - 0$

$\Rightarrow \aleph_0 = \#(S_1) + \aleph_0 \text{ ---- } ②$ نعوض في ①

$\alpha = \#(A_1) + \#(S_1) + \aleph_0$

if $x + y = z$, then ___ $\forall x, y, z \in \mathbb{N}$

$\alpha = \#(A_1 \cup S_1) + \aleph_0$

if $P(n+k) = PZ$, then ___

$\alpha = \#(A) + \aleph_0$

$\alpha = \alpha + \aleph_0$ ∴

The Natural Numbers ($\mathbb{N}$) اعداد العد

Definition:- The set $\mathbb{N}$ is said to be natural numbers if $\mathbb{N}$ satisfies the peano (بيانو) axioms (بديهيات):-

$2 \xrightarrow{\text{يتبعها}} 3$

3 هي n^+ لـ 2

1. $1 \in \mathbb{N}$.
2. if $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$. التابع
3. $\forall n \in \mathbb{N}$, $\exists n^+ \neq 1$. $n^+ = n+1$ لان
4. the Mathmatical Induction Axiom مبدأ الاستقراء الرياضي

   let $M \subseteq \mathbb{N}$ and M has the following properties:-

   a- $1 \in M$ b- if $n \in M$, then $n^+ \in M$.

   Then $M \equiv \mathbb{N}$.
5. $\forall n, m \in \mathbb{N}$, if $n^+ = m^+$, then $n = m$.

Addition of Natural Numbers:- نعرف عملية الجمع على الاعداد الطبيعية (خواص)

1. $n^+ = n+1$.
2. $n + m^+ = (n+m)^+$, $n + m^+ = n+m+1$.
3. $n+m = m+n$, $\forall n, m \in \mathbb{N}$. ابدالية
4. $\forall n, m, k \in \mathbb{N}$, $(n+m)+k = n+(m+k)$. تجميعية
5. if $n_1 + m = n_2 + m \Rightarrow n_1 = n_2$ قانون الحذف يتحقق لعملية الجمع

Amultiplication of Natural Numbers:-

1. $n \cdot 1 = n$.
2. $n m^+ = n(m+1) = nm + n$, $\forall n, m \in \mathbb{N}$.
3. $nm = mn$. ابدالية
4. $(nm)k = n(mk)$. تجميعية
5. $k(n+m) = kn + km$, $\forall n, m, k \in \mathbb{N}$ عملية الضرب تتوزع على عملية الجمع
6. $n_1 m = n_2 m \Rightarrow n_1 = n_2$

Lemma: let $n \in N$ and $n \neq 1$. Then $n > 1$ prove that

Proof: since $n \neq 1$, then $\exists\, m \in N$ such that $n = m' = m+1$

$\therefore m+1 > 1$

Hence $n > 1$

Theorem: The addition of natural numbers is unique binary operation.

Proof: let $\oplus$ be a binary operation defined as:

1. $n^+ = n \oplus 1$.
2. $n \oplus m^+ = (n \oplus m)'$

By using mathematical induction.

let $M = \{m \in N : n + m = n \oplus m,\ \forall n \in N\} \subseteq N$

a. $n^+ = n+1$ and $n^+ = n \oplus 1$, then $1 \in M$

b. let $m \in M$, that is $n + m = n \oplus m$, $\forall n \in N$

C. Now, to prove $m^+ \in M$.

$n + m^+ = (n+m)^+ = (n \oplus m)^+$ **by (2)**

$\therefore n + m^+ = n \oplus m^+$

$\therefore m^+ \in M$.

$\therefore M \equiv N$ (by P_4).

Theorem:- The Multiplication of natural numbers is unique binary operation.

Proof:- let $\otimes$ (direct product) be a binary operation defined as:-

1. $n^+ = n \otimes 1$.

2. $n \otimes m^+ = (n \otimes m)^+ = nm \otimes n$, $\forall n, m \in \mathbb{N}$.

By using mathematical induction:-

let $M = \{m \in \mathbb{N} : nm = n \otimes m, \forall n \in \mathbb{N}\} \subseteq \mathbb{N}$

a. $n^+ = n \cdot 1$ and $n^+ = n \otimes 1$, then $1 \in M$.

b. let $m \in M$, that is $n \times m = n \otimes m$, $\forall n \in \mathbb{N}$.

c. Now, to prove $m^+ \in M$.

$$n \times m^+ = n(m+1) = nm + n = n \otimes m + n = n \otimes (m+1)$$

$$\therefore n \times m^+ = n \otimes (m+1)$$

$$\therefore n \times m^+ = n \otimes m^+$$

$\Rightarrow m^+ \in M$.

$\Rightarrow M = \mathbb{N}$ (by P_4).

Def.:- If $a, b \in \mathbb{N}$, then $a < b$ iff $\exists\, k \in \mathbb{N}$ s.t. $a + k = b$.

if $x + y = z$, then $\underline{x < z}$ for all $x, z, y \in \mathbb{N}$

Lemma:- If $m < n$, then $m^+ \leq n$, $\forall\, n, m \in \mathbb{N}$

Proof:-

حسب التعريف

$\because m < n \Rightarrow \exists\, k \in \mathbb{N}$ s.t. $m + k = n$

If $k = 1$ then $m + 1 = n \Rightarrow m^+ = n$ --- ①

If $k > 1$, $\exists\, t \in \mathbb{N}$ s.t. $t + 1 = k$

تعويض عدد طبيعي اكبر

$\therefore n = m + k = m + (t+1) = m + (1+t) = (m+1) + t = m^+ + t$

$\therefore m^+ < n$ --- ②

From ① and ② we get $m^+ \leq n$.

Lemma:- let $n \in \mathbb{N}$. Then is not exists $y \in \mathbb{N}$ such that $n < y < n+1$

الأعداد الطبيعية بين كل عدد وعدد هو واحد

نفرض هذا صحيح

Proof:- suppose that $k \in \mathbb{N}$ s.t. $n < k < n+1$

1. if $n < k \Rightarrow$ there exist $k_1 \in \mathbb{N}$ s.t. $n + k_1 = k$.

نفرض عدد طبيعي

2. if $k < n+1 \Rightarrow$ there exist $k_2 \in \mathbb{N}$ s.t. $k + k_2 = n+1$

نعوض عن قيمة k

$\Rightarrow (n + k_1) + k_2 = n + 1$

$\Rightarrow \cancel{n} + (k_1 + k_2) = \cancel{n} + 1$ عملية الجمع تسمح بالحذف

$\Rightarrow k_1 + k_2 = 1$

$\Rightarrow k_1 < 1$ C! لأن أقل عدد طبيعي هو العدد واحد

لا يوجد عدد طبيعي أقل من 1

$\therefore n \nless y \nless n+1$

ما اكو عدد محصور بين n و $n+1$