**Exercises (2.1): (Homework)**

**(1)** For each of the following sequences, write a formula for the $n^{th}$ term and determine the limit (if it exists).

(*a*) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

(*b*) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

(*c*) $0.9, 0.99, 0.999, \ldots$

(*d*) $sin\frac{\pi}{2}, sin\ \pi, sin\frac{3\pi}{2}, sin\ 2\pi, sin\frac{5\pi}{2}, \ \ldots$

**(2)** Determine whether the sequence converges or diverges. If it converges, find the limit.

(*a*) $< \frac{1}{2}(1 + (-1)^{n+1}) >$

(*b*) $< ln\ \frac{n+1}{n} >$

(*c*) $< \frac{2^n}{3^{n+1}} >$

(*d*) $< \left(1 + \frac{2}{n}\right)^{\frac{1}{n}} >$

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# *Chapter Three*

# متسلسلات الأعداد الحقيقية

# *Series of Real Numbers*

**Definition (3.1): (Infinite Series)** المتسلسلة غير المنتهية

The **infinite series** of real numbers $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let $S_n$ denote the nth partial sum:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $< S_n >$ is **convergent**, i.e. $\lim_{n\to\infty} S_n = S$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent and we write $\sum_{n=1}^{\infty} a_n = S$.

The number $S$ is the **sum** of the series. Otherwise, the series is called **divergent**.

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**Definition (3.2): (Geometric Series)** المتسلسلة الهندسية

The series $\sum_{n=1}^{\infty} a\, r^{n-1} = a + ar + ar^2 + \cdots$ is called **geometric series** is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} a\, r^{n-1} = \frac{a}{1-r}\ , \quad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

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**Example (3.1):** Find the sum of the series

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Since $|r| = \frac{2}{3} < 1 \ \Rightarrow$ convergent

$$\Rightarrow \ \sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = \frac{5}{1-(-\frac{2}{3})} = 3$$

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**Definition (3.3): (Harmonic Series)** المتسلسلة التوافقية

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called **harmonic series** and its divergent.

**Theorem (3.1):** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

**Proof:** Let $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

$\Rightarrow \quad a_n = S_n - S_{n-1}$

Since $\sum_{n=1}^{\infty} a_n$ convergent

$\Rightarrow \; < S_n >$ is convergent

$\Rightarrow \lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_{n-1} = S$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

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**Remark (3.1):** The converse of the above theorem is not true as the following example.

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**Example (3.2):** Note that $\frac{1}{n} \to 0$ as $n \to \infty$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

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**Note (3.1):** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

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**Example (3.3):** Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$

**Solution:** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$

So the series diverges.

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**Theorem (3.2):** If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then

***(i)*** $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

***(ii)*** $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

***(ii)*** $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

**Definition (3.4): ($p$ - Series)**

The $\boldsymbol{p}$ **−series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

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**Example (3.4):**

**(1)** The series $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ is convergent because it is a p-series with $p = 3 > 1$

**(2)** The series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} \ldots$ is divergent because it is a p-series with $p = \frac{1}{3} < 1$

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**Theorem (3.3): (Comparison Test)** **اختبار المقارنة**

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms

**(i)** If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$, $\forall\, n$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

**(ii)** If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$, $\forall\, n$, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

**Proof:**

**(i)** Let $s_n = \sum_{i=1}^{n} a_i$ , $t_n = \sum_{i=1}^{n} b_i$ , $t = \sum_{n=1}^{\infty} b_n$

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have positive terms

$\Rightarrow$ $< s_n >$ and $< t_n >$ are increasing

Also, $t_n \to t$, so $t_n \leq t$, $\forall\, n$

Since $a_i \leq b_i$ , we have $s_n \leq t_n$

Thus, $s_n \leq t$, $\forall\, n$ (monotone bounded sequence)

$\Rightarrow$ $< s_n >$ converges

Therefore, $\sum_{n=1}^{\infty} a_n$ converges

**(ii)** If $\sum_{n=1}^{\infty} b_n$ is divergent, then $t_n \to \infty$

But $a_i \geq b_i$ so $s_n \geq t_n$

Thus, $s_n \to \infty$

Therefore, $\sum_{n=1}^{\infty} a_n$ diverges

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