Hence

(*i*) $T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$

$$= f(0) + \frac{f'(0)}{1!}(x-0)$$

$$= 0 + \frac{1}{1!}(x-0) = x$$

$$T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= 0 + \frac{1}{1!}(x-0) + \frac{0}{2!}(x-0)^2 = x$$

$$T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= 0 + \frac{1}{1!}(x-0) + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3 = x - \frac{x^3}{6}$$

(*ii*) $T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$

$$= f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right)$$

$$= 1 + \frac{0}{1!}\left(x-\frac{\pi}{2}\right) = 1$$

$$T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2$$

$$= 1 + \frac{0}{1!}\left(x-\frac{\pi}{2}\right) + \frac{-1}{2!}(x-\frac{\pi}{2})^2 = 1 - \frac{1}{2}(x-\frac{\pi}{2})^2$$

$$T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3$$

$$= 1 + \frac{0}{1!}\left(x - \frac{\pi}{2}\right) + \frac{-1}{2!}(x - \frac{\pi}{2})^2 + \frac{0}{3!}(x - \frac{\pi}{2})^3 = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$$

Theorem (1.9): (Taylor's Theorem)

Let f be $(n+1)$-times differentiable on an open interval containing the points a and x. Then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, and c is some point between a and x.

Corollary (1.6): (Remainder Estimate)

Let f be $(n+1)$-times differentiable on an open interval containing the points a and x. If $|f^{(n+1)}(c)| \leq M$, for all c between a and x, then

$$f(x) = T_n(x) + R_n(x)$$

Where

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Example (1.13): By applying the Remainder Estimate to the function $f(x) = sin\, x$, with $a = 0$ and $n = 3$, calculate $sin(0.1)$ to <u>four</u> decimal places.

Solution:

$f(x) = sin\, x,$ $\qquad$ $f(0) = 0,$

$f'(x) = cos\, x,$ $\qquad$ $f'(0) = 1,$

$f''(x) = -sin\, x,$ $\qquad$ $f''(0) = 0,$

$f'''(x) = -cos\, x,$ $\qquad$ $f'''(0) = -1,$

Hence the Taylor polynomial of degree 3 for f at 0 is

$T_3(x) = x - \frac{x^3}{6}$

Now,

$|R_3(0.1)| = \frac{|f^{(4)}(c)|}{(3+1)!}|0.1 - 0|^{3+1}$

$= \frac{|sin\ c|}{4!} \times (0.1)^4$

$\leq \frac{1}{24} \times (0.0001) = 0.041\bar{6} \times 10^{-4} < 0.5 \times 10^{-5}$

Now,

$sin(0.1) \simeq T_3(0.1) + R_3(0.1)$

$= \left(0.1 - \frac{0.001}{6}\right) + (0.5 \times 10^{-5})$

$= 0.0998333 \ldots + (0.5 \times 10^{-5}) = 0.0998$

Hence

$sin(0.1) = 0.0998$ (to <u>four</u> decimal places).

Example (1.14): By applying the Remainder Estimate to the function $f(x) = e^x$, with $a = 0$, calculate e to <u>three</u> decimal places.

Solution:

The Taylor polynomial of degree n for f at 0 is

$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$

$= f(0) + \frac{f'(0)}{1!}(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x - 0)^n$

$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$

Now,

$|R_n(1)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|1 - 0|^{n+1}$

$$= \frac{|e^c|}{(n+1)!} \times (1)^{n+1}$$

$$< \frac{3}{(n+1)!}, \quad for\ all \quad c \in (0,1)$$

To calculate e to three decimal places, we must choose n so that

$\frac{3}{(n+1)!} < 2 \times 10^{-4} \quad \Rightarrow \quad (n+1)! > 15{,}000$

Since $7! = 5{,}040$ and $8! = 40{,}320$, we may safely choose $n = 7$

It follows that

$$e \simeq T_7(1) + R_7(1)$$

$$\simeq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{3}{8!}$$

$$= 1 + 1 + 0.5 + 0.1\overline{6} + 0.041\overline{6} + 0.008\overline{3} + 0.0013\overline{8} + 0.000198412\ldots$$

$$+0.000074404\ldots$$

$$= 2.7182818\ldots$$

Hence, $e = 2.718$ (to three decimal places).

Example (1.15): Calculate the Taylor polynomial $T_3(x)$ for $f(x) = \frac{1}{x+2}$ at 1. Show that $T_3(x)$ approximates $f(x)$ with an error less than 5×10^{-3} on the interval [1,2].

Solution: Here

$$f(x) = \frac{1}{x+2}, \qquad f(1) = \frac{1}{3};$$

$$f'(x) = -\frac{1}{(x+2)^2}, \qquad f'(1) = -\frac{1}{9};$$

$$f''(x) = \frac{2}{(x+2)^3}, \qquad f''(1) = \frac{2}{27};$$

$$f'''(x) = -\frac{6}{(x+2)^4}, \qquad f'''(1) = -\frac{2}{27}.$$