The properties of Natural Numbers:-

خواص الاعداد الطبيعية

1. $n+m=m+n \quad \forall n,m\in N$ تبديلية
2. $(n+m)+p=n+(m+p) \quad \forall n,m,p\in N$ تجميعية
3. $nm=mn \quad \forall n,m\in N$ تبديلية
4. $(nm)p=n(mp) \quad \forall n,m,p\in N$ تجميعية
5. $n(m+p)=nm+np \quad \forall n,m,p\in N$
6. if $n+p=m+p \Rightarrow n=m \quad \forall n,m,p\in N$ قانون الحذف للجمع
7. if $np=mp \Rightarrow n=m \quad \forall n,m,p\in N$ قانون الحذف للضرب

Proof(1):- let $M=\{m\in N : n+m=m+n \;\; \forall n\in N\}\subseteq N$

to prove

1. $1\in M$, Now

   $n+1=n^{+}=1+n$ hence $1\in M$
2. let $m\in M$ i.e. $n+m=m+n$
3. to prove $m^{+}\in M$

   $\because n+m^{+}=(n+m)^{+}=(m+n)^{+}$

   $=1+(m+n)=(1+m)+n=m^{+}+n$

   $\therefore m^{+}\in M$ and by P_4 $\therefore M=N$

① مفروض المجموعة M جزئية من N

خاصية تجميعية

Proof (2) :- let $M=\{p\in N : (n+m)+p=n+(m+p) \;,\; \forall n,m,p\in N\} \subseteq N$.

to prove

استخدام الاستقراء الرياضي

1. $1\in M$, let $p=1$

$\therefore (n+m)+1=(n+m)^{+}=n+m^{+}=n+(m+1) \Rightarrow 1\in M$.

2. let $p\in M$

يعني متحققة

i.e. $(n+m)+p=n+(m+p)$.

3. to prove $p^{+}\in M$

Now, $(n+m)+p^{+}=[(n+m)+p]^{+}$ الكل أس by (2)

$=[n+(m+p)]^{+}$ لأن $p\in M$

$=n+(m+p)^{+}$

$=n+(m+p^{+}) \Rightarrow p^{+}\in M$

$\therefore M=N$ by (P_4).

Proof (4) :-

let $M=\{p\in N : (nm)p=n(mp) \;,\; n,m\in N\}$

1. If $p=1$, then $(nm)1=nm=n(m1) \Rightarrow 1\in M$. لأنها حققت المعادلة

2. suppose that $p\in M$ $\therefore (nm)p=n(mp)$.

3. Now, we prove $p^{+}\in M$

$(nm)p^{+}=nm(p+1)=nmp+nm$

$=n(mp+m)$

$=n(m(p+1))$ عامل مشترك

$=n(mp^{+})$

$\therefore p^{+}\in M$ and P_4 $M\equiv N$.

proof(6) :-

let $M = \{p \in N , \text{if } n+p = m+p \Rightarrow n = m , \forall n, m \in N\}$

1. to prove $1 \in M$, let $p = 1$

$\therefore n+1 = m+1 \Rightarrow n^+ = m^+ \Rightarrow n = m \Rightarrow 1 \in M$.

2. let $p \in M$ i.e. if $n+p = m+p \Rightarrow n = m$.

3. to prove $p^+ \in M$

if $n+p^+ = m+p^+$

$\Rightarrow (n+p)^+ = (m+p)^+$ اذا تساوت التوابع تساوت الاعداد

$\Rightarrow n+p = m+p$

from(2) $\Rightarrow n = m \Rightarrow p^+ \in M$.

$\therefore M \equiv N$ by P_4.

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proof (7) :- مجموعة M جزئية من N

let $M = \{p \in N : \text{if } np = mp \Rightarrow n = m , \forall n, m \in N\} \subseteq N$

1. to prove $1 \in M$

If $p = 1$ $\therefore n1 = m1 \Rightarrow n = m \Rightarrow 1 \in M$.

2. let $p \in M$ i.e. $np = mp \Rightarrow n = m$.

3. to prove $p^+ \in M$

if $np^+ = mp^+ \Rightarrow n(p+1) = m(p+1)$ نفتح p^+

$\Rightarrow np + n = mp + m$

$\Rightarrow n = m$ نفس البنية

حسب خاصية الحذف
لان عملية الحذف تخضع لعملية الجمع

$\therefore p^+ \in M \Rightarrow M \equiv N$ by P_4.

Theorem*:- If $n, m \in N$, then ~~بس واحد فقط بتحقق~~ exactly one of the following hold $m=n$ or $m<n$ or $m>n$.

بس واحد فقط بتحقق

Proof:- If m or n is equal to 1, then the theorem is true. (since $\forall n \neq 1 \Rightarrow n>1$, $\forall n \in N$).

Now, we must suppose that $n \neq 1$, $m \neq 1$, we use the Mathematical Induction.

let $S_1 = \{x \in N : x = n\}$

$S_2 = \{x \in N : x > n\}$

$S_3 = \{x \in N : x < n\}$

and let $S = S_1 \cup S_2 \cup S_3$, $S_1 \cap S_2 = \emptyset$ and $S_2 \cap S_3 = \emptyset$, $S_1 \cap S_3 = \emptyset$

Now, 1. $1 \in S$ since $\forall n \neq 1$, $n > 1 \Rightarrow 1 \in S_3 \subset S \Rightarrow 1 \in S$.

2. let $k \in S \Rightarrow k \in S_1$ or $k \in S_2$ or $k \in S_3$.

3. to prove $k^+ \in S$.

ⓐ let $k \in S_1 \Rightarrow k = n \Rightarrow k^+ = n^+ \Rightarrow k^+ = n+1 \Rightarrow k^+ > n$.
$\Rightarrow k^+ \in S_2 \subset S$.

ⓑ If $k \in S_2 \Rightarrow k > n \Rightarrow \exists \ell \in N$ s.t. $k = n + \ell$

Now, $k^+ = (n+\ell)^+ \overset{n+\ell^+}{=} n + (\ell+1) \Rightarrow k^+ > n \Rightarrow k^+ \in S_2 \subset S$.

ⓒ If $k \in S_3 \Rightarrow k < n \Rightarrow k^+ \leq n$ (by Lemma $\forall n, m \in N$, if $m<n$ then $m^+ \leq n$)

Now, if $k^+ = n \Rightarrow k^+ \in S_1 \subset S$

if $k^+ < n \Rightarrow k^+ \in S_3 \subset S$

$\therefore k^+ \in S$ $\quad$ $\therefore S \equiv N$.

$k < n \Rightarrow k + \ell = n$

$(k+\ell)^+ = n^+$

$k + \ell^+ = n^+$

Proof($\Rightarrow$): suppose that $n<m$, then

$\exists k \in N$ s.t. $n+k=m$ and $\exists p \in N$ s.t.

نضرب (الطرفين) ب p $\quad P(n+k)=pm \Rightarrow pn+pk=pm$

$$\Rightarrow pn<pm \quad (pk \in N)$$

$\Leftarrow$ suppose that $np<mp$, then

from theorem*(If $n,m \in N$, then exactly one of the following hold $m=n$ or $m<n$ or $m>n$)

we get

if $n=m \Rightarrow np=mp$ C!

if $n>m \Rightarrow np>mp$ C!

$\therefore n<m$.

if $p(n+k)=pz$, then ——

$pn+pk=pz$

$pn<pz$