Example (3.5): Test the series $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ for convergence or divergence.

Solutions: Since $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent and $\frac{\ln n}{n} > \frac{1}{n}$

$\Rightarrow \sum_{n=1}^{\infty}\frac{\ln n}{n}$ diverges

Definition (3.5): (Alternating Series) المتسلسلة المتناوبة

The **alternating series** $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, $(a_n > 0)$ is convergent if satisfies $a_{n+1} \leq a_n, \forall\, n$ and $\lim_{n\to\infty} a_n = 0$.

Example (3.6): Test the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ for convergence or divergence.

Solutions: Since $\sum_{n=1}^{\infty}\frac{1}{n}$ satisfies

$a_{n+1} \leq a_n$, because $\frac{1}{n+1} < \frac{1}{n}$, $\forall\, n$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0.$

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges

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Definition (3.6): (Absolutely Convergent) التقارب المطلق

A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Example (3.7): The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series $(p = 2)$

Definition (3.7): (Conditionally Convergent) **التقارب المشروط**

A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example (3.8): The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent, but it is not absolutely convergent because

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Which is the harmonic p-series ($p = 1$) and its divergent.

Theorem (3.4): If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Proposition (3.1): (Root Test) **اختبار الجذر**

Let $\sum_{n=1}^{\infty} a_n$ be infinite series, $a_n > 0, \ \forall\, n$, then

(i) If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L > 1$, or $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$, the Root Test is inconclusive.

Example (3.9): Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Solution: $a_n = \left(\frac{2n+3}{3n+2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1$$

Thus, the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converges.

Proposition (3.2): (Ratio Test) اختبار النسبة

Let $\sum_{n=1}^{\infty} a_n$ be infinite series, $a_n > 0,\ \forall\, n$, then

(i) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$, or $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the Ratio Test is inconclusive.

Example (3.10): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

Solution: $a_n = \frac{n^3}{3^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} . \frac{3^n}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 = \frac{1}{3} < 1$$

Thus, the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ is convergent.

Exercise (3.1): (Homework)

For each of the following series, determine which ones are convergent and which are divergent.

(1) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

(2) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$

(3) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(4) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

(5) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

Chapter Four
الفضاءات المترية
Metric Spaces

Definition (4.1): (Metric Space) الفضاء المتري

Let $X \neq \emptyset$ be a set and $d: X \times X \to R$ be a function satisfies that:

(1) $d(x,y) \geq 0,\ \forall\, x,y \in X$, $\quad d(x,y) = 0 \iff x = y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,z) \leq d(x,y) + d(y,z)$

Then d called a **metric** on X and (X,d) is a **metric space**.

Example (4.1): Let $X = R$, $d: R \times R \to R$ and $d(x,y) = |x - y|$. Show that (X,d) is a metric space.

Solution:

(1) $d(x,y) = |x - y| \geq 0$

$d(x,y) = 0 \iff |x - y| = 0 \iff x - y = 0 \iff x = y$

(2) $d(x,y) = |x - y| = |y - x| = d(y,x)$

(3) $d(x,z) = |x - z|$

$\leq |x - y| + |y - z|$

$= d(x,y) + d(y,z)$

$\Rightarrow d(x,z) \leq d(x,y) + d(y,z)$, $\quad \forall\ x,y,z \in R$

$\therefore$ (R,d) is a metric space or (d is a metric on R).

Example (4.2): Let $X \neq \emptyset$ be a set and $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. Show that d is a metric on X.

Solution:

(1) $d(x,y) \geq 0$, $\forall\ x,y \in X$