Hence the Taylor polynomial of degree 3 for f at 1 is

$$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= \frac{1}{3} + \frac{-1/9}{1!}(x-1) + \frac{2/27}{2!}(x-1)^2 + \frac{-2/27}{3!}(x-1)^3$$

$$= \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \frac{1}{81}(x-1)^3$$

Now,

$$|R_3(2)| = \frac{|f^{(4)}(c)|}{(3+1)!}|2-1|^{3+1}$$

$$= \frac{\left|\frac{24}{(x+2)^5}\right|}{4!} \times (1)^4$$

$$< \frac{\left|\frac{24}{(1+2)^5}\right|}{24} = \frac{\frac{24}{3^5}}{24} < \frac{1}{3^5} = 0.00041 \ldots, \quad for \quad x \in [1,2]$$

Since $|R_3(x)| < 0.00041 < 5 \times 10^{-3}$, $for \quad x \in [1,2]$

It follows that $T_3(x)$ approximates $f(x)$ with an error less than 5×10^{-3} on $[1,2]$.

Example (1.16): Calculate $T_1(x)$ and $R_1(x)$ for the function

$$f(x) = log_e(1+x), \ x \in (-1,1), \text{ at } 0.$$

Solution:

For the function $f(x) = log_e(1+x), \ x \in (-1,1)$, we have

$f(x) = log_e(1+x)$, $\qquad f(0) = 0$;

$f'(x) = \dfrac{1}{1+x}$, $\qquad f'(0) = 1$;

$f''(x) = \dfrac{-1}{(1+x)^2}$, $\qquad f''(0) = -1$;

Hence

$T_1(x) = f(0) + f'(0)x = x$ and $R_1(x) = \frac{f''(c)}{2!}x^2 = \frac{-x^2}{2(1+c)^2}$, for some number c between 0 and x .

Theorem (1.10): (Basic Power Series)

$$(1)\ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad = \sum_{n=0}^{\infty} x^n , \qquad for\ |x| < 1$$

$$(2)\ log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=0}^{\infty} (-1)^{n+1}\frac{x^n}{n}, \qquad for\ |x| < 1 ;$$

$$(3)\ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad for\ x \in R ;$$

$$(4)\ sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad for\ x \in R ;$$

$$(5)\ cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad for\ x \in R .$$

Exercise (1.3) (Homework)

(1) Determine the Taylor polynomials $T_1(x)$, $T_2(x)$ and $T_3(x)$ for each of the following functions f at given point a:

(a) $f(x) = e^x , a = 2$;

(b) $f(x) = cos\ x , a = 0$.

(2) Determine the Taylor polynomial of degree 4 for each of the following functions f at the given point a:

(a) $f(x) = 7 - 6x + 5x^2 + x^3, a = 1$;

(b) $f(x) = \frac{1}{1-x}, a = 0$;

(c) $f(x) = log_e(1+x) , a = 0$;

(d) $f(x) = sin\, x$, $a = \frac{\pi}{4}$;

(e) $f(x) = 1 + \frac{1}{2}x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{4}x^4$, $a = 0$.

Chapter Two
التكامل الريماني
Riemann Integration

Definition (2.1): (Partition) التجزئة

A **partition** P of an interval $[a, b]$ is a family of a finite number of subintervals of $[a, b]$

$$P = \{[x_0, x_1], [x_1, x_2], \dots , [x_{i-1}, x_i], \dots , [x_{n-1}, x_n]\}$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \ \dots < x_{n-1} < x_n = b$$

The points x_i, $0 \le i \le n$, are called the **partition points** in P.

معيار

The **length of the ith subinterval** is denoted by $\delta x_i = x_i - x_{i-1}$, and the **norm (**or **mesh)** of P is the quantity $\|P\| = \max\limits_{1 \le i \le n}\{\delta x_i\}$.

قياسية تجزئة

A **standard partition** is a partition with <u>equal</u> subintervals.

Example (2.1): Consider the partition P of $[0,1]$, where

$$P = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{5}\right], \left[\frac{3}{5}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]\right\}$$

Here

$\delta x_1 = \frac{1}{2} - 0 = \frac{1}{2}$,, $x_2 = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$, $\delta x_2 = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$ and $\delta x_2 = 1 - \frac{3}{4} = \frac{1}{4}$

and the **norm (**or **mesh)** of P is

$\|P\| = max\left\{\frac{1}{2}, \frac{1}{10}, \frac{3}{20}, \frac{1}{4}\right\} = \frac{1}{2}$

P is not a standard partition of $[0,1]$, since not all its subintervals are of equal length.
