$d(x,y) = 0 \iff x = y$

(2) If $x \neq y \Rightarrow d(x,y) = 1 = d(y,x)$

If $x = y \Rightarrow d(x,y) = 0 = d(y,x)$

(3) If $x = y, y = z \Rightarrow d(x,y) + d(y,z) = 0 + 0 = 0 = d(x,z)$

If $x = y, y \neq z \Rightarrow d(x,y) + d(y,z) = 0 + 1 = 1 = d(x,z)$

If $x \neq y, y = z \Rightarrow d(x,y) + d(y,z) = 1 + 0 = 1 = d(x,z)$

If $x \neq y,\ y \neq z \Rightarrow d(x,y) + d(y,z) = 1 + 1 = 2 > 1 = d(x,z)$

$\therefore\ d(x,z) \leq d(x,y) + d(y,z)\ ,\ \forall\, x,y,z \in X$

$\therefore$ (X,d) is a metric space

Example (4.3): Let $X \neq \emptyset$ be a set and $d(x,y) = |x^2 - y^2|$. Determine whether d is a metric on X or not.

Solution:

(1) $d(x,y) = |x^2 - y^2| \geq 0, \forall\ x,y \in X$

$d(x,y) = 0 \iff |x^2 - y^2| = 0$

$\iff x^2 - y^2 = 0$

$\iff x^2 = y^2 \Rightarrow x = y$ or $x = -y$

$\therefore$ (X,d) is not a metric space.

Example (4.4): Let $X \neq \emptyset$ be a set and $d(x,y) = |x - 2y|$. Determine whether d is a metric on X or not.

Solution:

(1) $d(x,y) = |x - 2y| \geq 0, \forall\ x,y \in X$

$d(x,y) = 0 \iff |x - 2y| = 0$

$\iff x - 2y = 0$

$\iff x = 2y \Rightarrow x \neq y$

$\therefore$ (X,d) is not a metric space.

Example (4.5): Let $X = R$ and $d(x,y) = \sqrt{|x-y|}$. Show that (X,d) is a metric space.

Solution:

(1) $d(x,y) = \sqrt{|x-y|} \geq 0$

$d(x,y) = 0 \Leftrightarrow \sqrt{|x-y|} = 0 \Leftrightarrow |x-y| = 0 \Leftrightarrow x-y = 0 \Leftrightarrow x = y$

(2) $d(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d(y,x)$

(3) $d(x,z) = \sqrt{|x-z|}$

$= \sqrt{|x-y+y-z|}$

$\leq \sqrt{|x-y|+|y-z|}$

$\leq \sqrt{|x-y|} + \sqrt{|y-z|}$

$= d(x,y) + d(y,z)$

$\therefore\ d(x,z) \leq d(x,y) + d(y,z)\ , \quad \forall\ x,y,z \in R$

$\therefore\ (R,d)$ is a metric space or (d is a metric on R).

Example (4.6): Let $X = R^2 = \{x = (x_1,x_2): x_i \in R\}$ and $d: R^2 \times R^2 \to R$ defined by $d(x,y) = [\sum_{i=1}^{2}(x_i - y_i)^2]^{\frac{1}{2}}$. Show that (X,d) is a metric space.

(1) $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} \geq 0\ , \quad \forall\, x,y \in R^2$

$d(x,y) = 0 \Leftrightarrow \left(\sum_{i=1}^{2}(x_i-y_i)^2\right)^{\frac{1}{2}} = 0$

$\Leftrightarrow \sum_{i=1}^{2}(x_i-y_i)^2 = 0$

$\Leftrightarrow x_i - y_i = 0 \Leftrightarrow x_i = y_i\ , \quad \forall\ i = 1,2$

(2) $d(x,y) = [\sum_{i=1}^{2}(x_i-y_i)^2]^{\frac{1}{2}}$

$= [\sum_{i=1}^{2}(y_i-x_i)^2]^{\frac{1}{2}}$

$= d(y,x)$

(3) $d(x,z) = [\sum_{i=1}^{2}(x_i-z_i)^2]^{\frac{1}{2}}$

$= \sqrt{(x_1-z_1)^2 + (x_2-z_2)^2}$

$$= \sqrt{(x_1 - y_1 + y_1 - z_1)^2 + (x_2 - y_2 + y_2 - z_2)^2}$$

$$\leq \sqrt{(x_1 - y_1)^2 + (y_1 - z_1)^2 + (x_2 - y_2)^2 + (y_2 - z_2)^2}$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} + \sqrt{(y_1 - z_1)^2 + (y_2 - z_2)^2}$$

$$= [\sum_{i=1}^{2}(x_i - y_i)^2]^{\frac{1}{2}} + [\sum_{i=1}^{2}(y_i - z_i)^2]^{\frac{1}{2}}$$

$$= d(x, y) + d(y, z), \quad \forall\ x, y, z \in R^2$$

$\therefore$ (R^2, d) is a metric space

Example (4.7): Let (X, d) be a metric space and $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Show that (X, ρ) is a metric space.

(1) Since $d(x, y) > 0 \Rightarrow \frac{d(x,y)}{1+d(x,y)} > 0 \Rightarrow \rho(x, y) > 0$

Since $d(x, y) = 0 \Leftrightarrow x = y$

$\Rightarrow \rho(x, y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0 \Leftrightarrow x = y$

(2) Since $d(x, y) = d(y, x)$

$\Rightarrow \rho(x, y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \rho(y, x)$

(3) We have

$d(x, z) \leq d(x, y) + d(y, z)$

$\Rightarrow \frac{1}{d(x,z)} \geq \frac{1}{d(x,y)+d(y,z)}$

$\Rightarrow 1 + \frac{1}{d(x,z)} \geq 1 + \frac{1}{d(x,y)+d(y,z)}$

$\Rightarrow \frac{1+d(x,z)}{d(x,z)} \geq \frac{1+d(x,y)+d(y,z)}{d(x,y)+d(y,z)}$

$\Rightarrow \frac{d(x,z)}{1+d(x,z)} \leq \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$

$\Rightarrow \frac{d(x,z)}{1+d(x,z)} \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$

$\Rightarrow \rho(x, z) \leq \rho(x, y) + \rho(y, z)$

$\therefore$ (X, ρ) is a metric space

Definition (4.2): Let (X, d) be a metric space and let $x_0 \in X$, $r \geq 0$. Then

(1) $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$ a set called the **ball** of center x_0 and radius r.

(2) $S_r(x_0) = \{x \in X : d(x, x_0) = r\}$ a set called the **sphere** of center x_0 and radius r

(3) $D_r(x_0) = \{x \in X : d(x, x_0) \leq r\}$ a set called the **disk** of center x_0 and radius r.

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Note (4.1): Sometimes we write $B_r(x_0)$ as $N_\varepsilon(x_0)$ and read the **neighborhood (nbh)** of a point $x_0 \in X$, i.e. $N_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}$.

Definition (4.3): (Interior Set) المجموعة الداخلية

Let (X, d) be a metric space and $A \subset X$, the point x is an **interior point** of A iff $\exists\, r > 0$ s.t. $B_r(x) \subseteq A$. The set of <u>all</u> interior points is called **interior set** of A and denoted by A°.

Example (4.8): Find the interior set for each of the following sets

(1) $A = \{1,2,3,4\}$,

Since $\forall\, x \in A$, $B_r(x) \nsubseteq A$, $\forall\, r > 0$

$\Rightarrow\ A^\circ = \emptyset$

(2) $A = \{x \in R : 1 < x \leq 3\}$

Since $B_r(3) \nsubseteq A$, $\forall\, r > 0 \ \Rightarrow$ 3 not an interior point of A

Since $\forall\, x \in (1,3) \ \Rightarrow\ B_r(x) \subseteq A,\ r > 0$

$\Rightarrow\ A^\circ = (1,3)$

(3) $A = \{x \in R : -1 < x < 1\}$

Since $\forall\, x \in (-1,1) \ \Rightarrow\ B_r(x) \subseteq A,\ r > 0$

$\Rightarrow\ A^\circ = (-1,1)$

(4) $A = \{x \in R : 0 \leq x\}$

Since $B_r(0) \nsubseteq A$, $\forall\, r > 0 \ \Rightarrow$ 0 not an interior point of A

Since $\forall\, x \in (0, \infty) \ \Rightarrow\ B_r(x) \subseteq A,\ r > 0$

$\Rightarrow\ A^\circ = (0, \infty)$