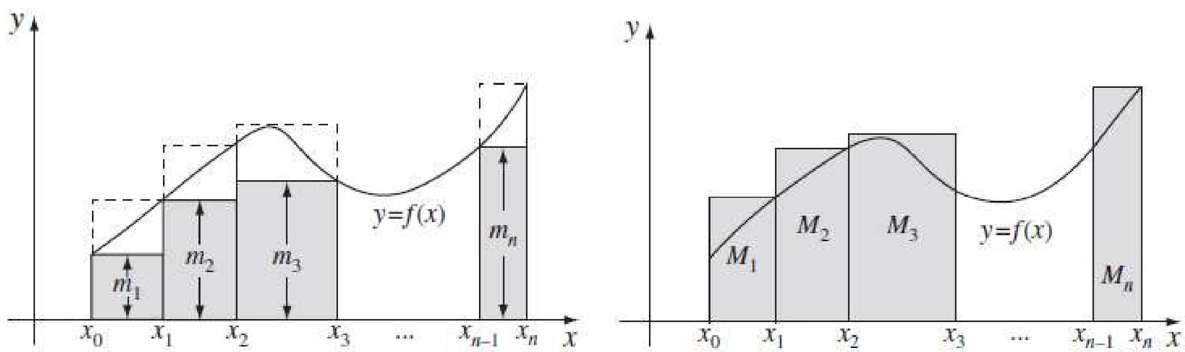
Definition (2.2): (Lower and Upper Riemann Sums) مجاميع ريمان العليا والسفلى

Let f be a bounded function on $[a,b]$, and P a partition of $[a,b]$ given by $P=\{[x_{i-1},x_i] : 1 \le i \le n\}$. We denote by m_i and M_i the quantities

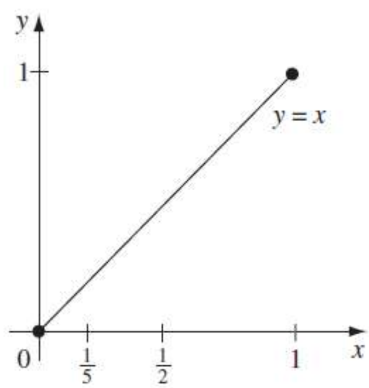
$$m_i = inf\{f(x): x \in [x_{i-1},x_i]\} \text{ and } M_i = sup\{f(x): x \in [x_{i-1},x_i]\}.$$

Then the corresponding **lower** and **upper Riemann sums** for f on $[a,b]$ are

$$L(f,P)=\sum_{i=1}^{n} m_i\delta x_i \quad \text{and} \quad U(f,P)=\sum_{i=1}^{n} M_i\delta x_i$$



Example (2.2): Let $f(x)=x$, $x \in [0,1]$, and let $P=\left\{\left[0,\frac{1}{5}\right],\left[\frac{1}{5},\frac{1}{2}\right],\left[\frac{1}{2},1\right]\right\}$ be a partition of $[0,1]$. Evaluate $L(f,P)$ and $U(f,P)$.



Solution:

Since f is increasing and continuous.

The infimum (inf) of f is the value of f at the left end-point of the subinterval and

The supremum (sup) of f is the value of f at the right end-point of the subinterval.

Hence,

$m_1 = f(0) = 0$, $M_1 = f\left(\frac{1}{5}\right) = \frac{1}{5}$, $\delta x_1 = \frac{1}{5} - 0 = \frac{1}{5}$,

$m_2 = f\left(\frac{1}{5}\right) = \frac{1}{5}$, $M_2 = f\left(\frac{1}{2}\right) = \frac{1}{2}$, $\delta x_2 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$,

$m_3 = f\left(\frac{1}{2}\right) = \frac{1}{2}$, $M_2 = f(1) = 1$, $\delta x_3 = 1 - \frac{1}{2} = \frac{1}{2}$.

It follows that

$$L(f,P) = \sum_{i=1}^{3} m_i \delta x_i = m_1\delta x_1 + m_2\delta x_2 + m_3\delta x_3$$

$$= 0 \times \frac{1}{5} + \frac{1}{5} \times \frac{3}{10} + \frac{1}{2} \times \frac{1}{2}$$

$$= 0 + \frac{3}{50} + \frac{1}{4} = \frac{31}{100}$$

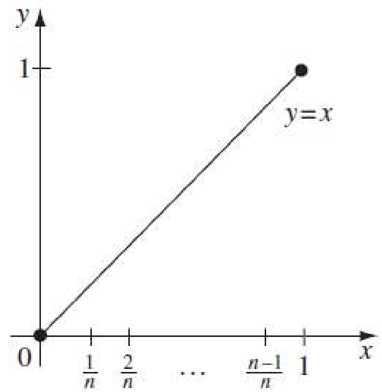
$$U(f,P) = \sum_{i=1}^{3} M_i \delta x_i = M_1\delta x_1 + M_2\delta x_2 + M_3\delta x_3$$

$$= \frac{1}{5} \times \frac{1}{5} + \frac{1}{2} \times \frac{3}{10} + 1 \times \frac{1}{2}$$

$$= \frac{1}{25} + \frac{3}{20} + \frac{1}{2} = \frac{69}{100}$$

Example (2.3): Evaluate $L(f, P_n)$ and $U(f, P_n)$ for the following function and standard partition of [0,1], $f(x) = x$, $x \in [0,1]$, and

$P_n = \left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{i-1}{n}, \frac{i}{n}\right], \dots, \left[1 - \frac{1}{n}, 1\right]\right\}$; and determine $\lim_{n\to\infty} L(f, P_n)$ and $\lim_{n\to\infty} U(f, P_n)$, if these exist.



Solution:

Since f is increasing and continuous.

The infimum (inf) of f is the value of f at the left end-point of the subinterval and

The supremum (sup) of f is the value of f at the right end-point of the subinterval.

Hence, on the ith subinterval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ in P_n, for $1 \le i \le n$, we have

$m_i = f\left(\frac{i-1}{n}\right) = \frac{i-1}{n}$, $M_1 = f\left(\frac{i}{n}\right) = \frac{i}{n}$, $\delta x_1 = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}$,

It follows that

$$L(f, P_n) = \sum_{i=1}^{n} m_i \delta x_i = \sum_{i=1}^{n} \frac{i-1}{n} \times \frac{1}{n}$$

$$= \frac{1}{n^2}\{\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\}$$

$$= \frac{1}{n^2}\left\{\frac{n(n+1)}{2} - n\right\} = \frac{n-1}{2n}$$

$$U(f, P_n) = \sum_{i=1}^{n} M_i \delta x_i = \sum_{i=1}^{n} \frac{i}{n} \times \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} i$$

$$= \frac{1}{n^2} \times \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

It follows that

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

Theorem (2.1): For any function f bounded on an interval $[a,b]$ and any partition P of $[a,b]$, $L(f,P) \leq U(f,P)$.

Proof:

Let $P = \{[x_{i-1}, x_i] : 1 \leq i \leq n\}$

Then, we have

$$inf\{f(x): x \in [x_{i-1}, x_i]\} \leq sup\{f(x): x \in [x_{i-1}, x_i]\}, \quad \forall\, [x_{i-1}, x_i], 1 \leq i \leq n$$

$$\Rightarrow\ m_i \leq M_i$$

$$\Rightarrow \sum_{i=1}^{n} m_i \delta x_i \leq \sum_{i=1}^{n} M_i \delta x_i$$

$$\therefore\ L(f,P) \leq U(f,U)$$

Definition (2.3): (Refinement) التنعيم

The partition P' is a **refinement** of P if $P' \supset P$ (that is, if every point of P is a point of P'). Given two partitions, P_1 and P_2, we say that P' is their **common refinement** if $P' = P_1 \cup P_2$.

Example (2.4): The partition $P' = \left\{\left[0,\frac{1}{2}\right], \left[\frac{1}{2},\frac{3}{5}\right], \left[\frac{3}{5},\frac{3}{4}\right], \left[\frac{3}{4},1\right]\right\}$ of $[0,1]$ is a **refinement** of the partition $P = \left\{\left[0,\frac{1}{2}\right], \left[\frac{1}{2},\frac{3}{4}\right], \left[\frac{3}{4},1\right]\right\}$, since it simply has one additional partition point $\frac{3}{5}$ as compared with P.
