③ $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$

Proof:- let $(x,y)\in(A\times B)\cap(C\times D)$

$\Rightarrow (x,y)\in(A\times B)$ and $(x,y)\in(C\times D)$

$\Rightarrow (x\in A$ and $y\in B)$ and $(x\in C$ and $y\in D)$

$\Rightarrow (x\in A$ and $x\in C)$ and $(y\in B$ and $y\in D)$

$\Rightarrow x\in(A\cap C)$ and $y\in(B\cap D)$

$\Rightarrow (x,y)\in(A\cap C)\times(B\cap D)$

$\therefore (A\times B)\cap(C\times D)\subseteq(A\cap C)\times(B\cap D)$ --- ①

$(A\cap C)\times(B\cap D)\subseteq(A\times B)\cap(C\times D)$ --- ②

from (1) and (2) we get

$(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$.

⑥ $(A\times A)-(B\times C)=[(A-B)\times A]\cup[A\times(A-C)]$

Proof:- let $(x,y)\in[(A-B)\times A]\cup[A\times(A-C)] \leftarrow$ نختار الطرف الأعقد

$\Rightarrow (x,y)\in[(A-B)\times A]$ or $(x,y)\in[A\times(A-C)]$

$\Rightarrow (x\in(A-B)$ and $y\in A)$ or $(x\in A$ and $y\in(A-C))$

$\Rightarrow (x\in(A\cap B^c)$ and $y\in A)$ or $(x\in A$ and $y\in(A\cap C^c))$

$\Rightarrow (x\in A$ and $x\in B^c$ and $y\in A)$ or $(x\in A$ and $y\in A$ and $y\in C^c)$

$\Rightarrow (x\in A$ and $x\notin B$ and $y\in A)$ or $(x\in A$ and $y\in A$ and $y\notin C)$

$\Rightarrow (x\in A$ and $y\in A)$ or $(x\notin B$ and $y\notin C)$

$\Rightarrow [(x,y)\in A\times A]$ and $[(x,y)\notin B\times C]$

$\Rightarrow (x,y)\in(A\times A)-(B\times C)$.

$\therefore [(A-B)\times A]\cup[A\times(A-C)]\subseteq(A\times A)-(B\times C)$ --- (a)

$(A\times A)-(B\times C)\subseteq[(A-B)\times A]\cup[A\times(A-C)]$ --- (b)

From (a) and (b) we get

$(A\times A)-(B\times C)=[(A-B)\times A]\cup[A\times(A-C)]$.

Note: ① $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) : x_1 \in A_1, \ldots, x_n \in A_n\}$

② $A \times B = \varnothing$ iff: $A = \varnothing$ or $B = \varnothing$.

③ $(A \times B) \cap (A^c \cap C) = \varnothing$.

④ $A \times B = C \times D$ iff $A = C$ and $B = D$.

Ex.:- $U = \{1, 2, 6\}$

$A = \{1, 2\}$

$B = \{3, 4\}$

$C = \{5\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$A^c = \{6\}$

$A^c \cap C = \varnothing$

$\therefore (A \times B) \cap (A^c \cap C) = \varnothing$.

Relations

تعريف العلاقة

Def.1- Let A be a set, be a relation in A we mean an arbitrary subset of $A \times A$,

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$$R = \{(x,y) : x \in A \text{ and } y \in A\}$$

$$(x,y) \in R \text{ or } xRy$$

Example:- let $A = \{1,2,3\}$

$$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

$R_1 : x = y = \{(1,1),(2,2),(3,3)\}$

$R_2 : x < y = \{(1,2),(1,3),(2,3)\}$

Def.:- If R is a relation, then R^{-1} is a relation defined by

$$R^{-1} = \{(x,y) : (y,x) \in R\}$$

$$(x,y) \in R^{-1} \Leftrightarrow (y,x) \in R$$

Ex.:- let $A = \{a,b,c\}$

$R = \{(a,b),(b,c)\}$

$R^{-1} = \{(b,a),(c,b)\}$

Th.:- Let R be a relation in A. Then $(R^{-1})^{-1} = R$.

Proof:- let $(x,y) \in (R^{-1})^{-1}$

$\Rightarrow (y,x) \in R^{-1}$

$\Rightarrow (x,y) \in R$

$\Rightarrow (R^{-1})^{-1} \subseteq R$ ---- (a)

Now, let $(a,b) \in R$

$\Rightarrow (b,a) \in R^{-1}$

$\Rightarrow (a,b) \in (R^{-1})^{-1}$

$\Rightarrow R \subseteq (R^{-1})^{-1}$ ------ (b)

from (a) and (b) we get $(R^{-1})^{-1} = R$.

Ex.:- let $R_1 = \{(x,y) \in N \times N : x \leq y\}$

$R_2 = \{(x,y) \in N \times N : x \geq y\}$

then find ① $R_1 \cap R_2$ ② $R_1 \cup R_2$ ③ $R_1 - R_2$.

Sol.:- اولاً نكتب R_1 و R_2 بشكل عناصر

$R_1 = \{(1,1),(2,2),(3,3), \dots, (n,n), (1,2),(2,3), \dots\}$

$R_2 = \{(1,1),(2,2),(3,3), \dots, (n,n), (2,1),(3,2), \dots\}$

① $R_1 \cap R_2 = \{(x,y) \in N \times N : x = y\}$

$\therefore R_1 \cap R_2 = \{(1,1),(2,2),(3,3), \dots\}$

② $R_1 \cup R_2 = \{(x,y) \in N \times N : x \leq y\} \cup \{(x,y) \in N \times N : x \geq y\}$

$\therefore R_1 \cup R_2 = \{(1,1),(2,2),(3,3), \dots, (1,2),(1,3), \dots, (2,1),(3,1), \dots\}$

③ $R_1 - R_2 = \{(x,y) \in N \times N : x < y\}$

$= \{(1,2),(1,3),(2,3), \dots\}$

توضيح:

$R_1 - R_2 = R_1 \cap R_2^c$

$= (x \leq y) \cap (x < y)$

$= (x < y)$