
Definition (4.4): (Open Set) **المجموعة المفتوحة**

Let (X, d) be a metric space and $A \subseteq X$, then A is said to be **open set** if every point of A is an interior point, i.e. $A = A^\circ$.

Example (4.9):

(1) $A = \{x \in R : -1 < x < 1\}$

Since $A^\circ = (-1,1) = A \Rightarrow A$ is an open set.

(2) $A = \{x \in R : 0 \leq x\}$

Since $A^\circ = (0, \infty) \neq A \Rightarrow A$ is not an open set.

(3) $A = R$

Since $R^\circ = (-\infty, \infty) = R \Rightarrow R$ is an open set.

(4) $A = Q$

Since $Q^\circ = \emptyset \neq Q \Rightarrow Q$ is not an open set.

Definition (4.5): (Closed Set) **المجموعة المغلقة**

Let (X, d) be a metric space and $A \subseteq X$, then A is said to be **closed set** if $X - A$ is open set.

Example (4.10):

(1) $A = \{x \in R : -1 < x < 1\} \Rightarrow A$ is not a closed set.

(2) $A = \{x \in R : 0 \leq x\} \Rightarrow A$ is a closed set.

(3) $A = R \Rightarrow R$ is a closed set.

(4) $A = Q \Rightarrow Q$ is not a closed set.

Note (4.2): A subset which is both open and closed is called **Clopen set.**

Remark (4.1): Every open ball is open set.

Proof: Let $B_r(a)$ be an open ball.

Let $x \in B_r(a)$ and $\varepsilon = r - d(x,a)$

If $y \in B_\varepsilon(x)$

$\Rightarrow \; d(y,a) \leq d(y,x) + d(x,a) < \varepsilon + d(x,a) = r$

Thus, $y \in B_r(a)$

$\Rightarrow \; B_\varepsilon(x) \subseteq B_r(a)$.

Theorem (4.2): Let (X,d) be a metric space, then:

(1) The sets $\emptyset$ and X are both open.

(2) Any union of open sets is open.

(3) Finite intersection of open sets is open.

Proof:

(1) Since $\emptyset$ contains no points, the result follows

$\Rightarrow \; \emptyset$ is open

Now, since $\forall\, x \in X, \exists\, \varepsilon > 0$, s.t. $B_\varepsilon(x) \subseteq X$,

$\Rightarrow \; X$ is open

(2) Let $\{G_\alpha\}$ be a family of open sets

Let $x \in \bigcup_\alpha G_\alpha$

$\Rightarrow x \in G_\alpha$ for some α

Since G_α is open

$\Rightarrow \; \exists\, \varepsilon > 0$, s.t. $B_\varepsilon(x) \subseteq G_\alpha$

$\Rightarrow \; B_\varepsilon(x) \subseteq \bigcup_\alpha G_\alpha$

$\therefore \; \bigcup_\alpha G_\alpha$ is open set.

(3) Let $\{G_i, 1 \leq i \leq n\}$ be a family of open sets

Let $x \in \bigcap_{i=1}^{n} G_i$

$\Rightarrow x \in G_i \, , \forall\, i$

Since G_i is open $\forall\, i$

$\Rightarrow\ \exists\ \varepsilon > 0$, s.t. $B_\varepsilon(x) \subseteq G_i$, $\forall\ i$

$\Rightarrow\ B_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} G_i$

$\therefore\ \bigcap_{i=1}^{n} G_i$ is open set.

Note (4.3): Infinite intersection of open sets in metric space may not be open.

Example (4.11): Let $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in N$ be a family of open sets

$G_1 = (-1,1)$, $G_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$, $G_3 = \left(-\frac{1}{3}, \frac{1}{3}\right)$, … , $G_\infty = \left(-\frac{1}{\infty}, \frac{1}{\infty}\right) = \{0\}$

$\Rightarrow\ \bigcap_{n=1}^{\infty} G_n = \{0\}$ which is not open

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Definition (4.5): (Limit Point) **نقطة الغاية**

Let (X, d) be a metric space. If $A \subset X$, we say that the point $x \in X$ is a **limit point** of A iff $\forall\ B_\varepsilon(x) \ni x$; $(B_\varepsilon(x) \cap A) \backslash \{x\} \neq \emptyset$, for any $\varepsilon > 0$.

Definition (4.6): (Derived Set) **المجموعة المشتقة**

Let (X, d) be a metric space and $A \subseteq X$, The set of <u>all</u> limit points in A is called **derived set** of A and denoted by A'. If $A' \subseteq A$ then A is a closed set.

Example (4.12): Let (R, d) be a metric space and $A = [2,3)$

Since when $x = 2$, we have $\forall\ B_\varepsilon(2) \ni 2$; $(B_\varepsilon(2) \cap [2,3)) \backslash \{2\} \neq \emptyset$

$\Rightarrow$ 2 is a limit point

and when $x = 3$, we have $\forall\ B_\varepsilon(3) \ni 3$; $(B_\varepsilon(3) \cap [2,3)) \backslash \{3\} \neq \emptyset$

$\Rightarrow$ 3 is a limit point

$\therefore\ A' = [2,3]$

Example (4.13):

(1) $A = R$ is a closed set

Since $\forall\ x \in R$ and $\forall\ B_\varepsilon(x) \ni x$; $(B_\varepsilon(x) \cap R) \backslash \{x\} \neq \emptyset$.

$\Rightarrow\ R' = (-\infty, \infty) = R$

(2) Q is not a closed set

Since $Q \in R$, and $\forall\ x \in R$, $\forall\ B_\varepsilon(x) \ni x$; $(B_\varepsilon(x) \cap Q)\backslash\{x\} \neq \emptyset$.

Thus, $Q' = (-\infty, \infty) = R \nsubseteq Q$

Theorem (4.3): Let (X, d) be a metric space, then:

(1) The sets $\emptyset$ and X are both closed.

(2) Any intersection of closed sets is closed.

(3) Finite union of closed sets is closed.

Proof:

(1) Since X is open

$\Rightarrow\ X^c = \emptyset$ is closed

And since $\emptyset$ is open

$\Rightarrow\ \emptyset^c = X$ is closed

(2) Let $\{F_\alpha\}$ be a family of closed sets

Since F_α is closed , $\forall\ \alpha$

$\Rightarrow\ F_\alpha^c$ is open , $\forall\ \alpha$

$\Rightarrow\ \bigcup_\alpha F_\alpha^c$ is open set.

$\Rightarrow\ \left(\bigcup_\alpha F_\alpha^c\right)^c = \bigcap_\alpha F_\alpha$ is closed set

(3) Let $\{F_i, 1 \leq i \leq n\}$ be a family of closed sets

Since F_i is closed , $\forall\ i$

$\Rightarrow\ F_i^c$ is open , $\forall\ i$

$\Rightarrow\ \bigcap_{i=1}^n F_i^c$ is open set.

$\Rightarrow\ (\bigcap_{i=1}^n F_i^c)^c = \bigcup_{i=1}^n F_i$ is closed set.

Definition (4.7): (Closure of a Set) **انغلاق المجموعة**

Let (X, d) be a metric space and $A \subset X$, the **closure** of a set A, is defined as: $\bar{A} = A \cup A'$.