Lemma (2.1): Let f be a bounded function on an interval $[a,b]$. Let P and P' be partitions of $[a,b]$, where P' is a refinement of P that contains just one additional partition point. Then

$$L(f,P) \leq L(f,P') \quad \text{and} \quad U(f,P') \leq U(f,P).$$

Definition (2.4): (Riemann-Integrable Function) دالـة قـابلـة للـتـكـامـل الريـمـانـي

Let f be a bounded function on an interval $[a,b]$. Then we define:

- The **lower Riemann integral of *f*** on $[a,b]$ to be $\underline{\int_a^b} f = \sup_P L(f,P)$,
- The **upper Riemann integral of *f*** on $[a,b]$ to be $\overline{\int_a^b} f = \inf_P U(f,P)$,

Where P denotes partitions of $[a,b]$.

Further, if the lower and upper integrals are equal, we say that f is **Riemann-integrable** on $[a,b]$, ($f \in \mathcal{R}[a,b]$) or $\int_a^b f$ to be their common value; that is

$$\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$$

Example (2.5): Prove that the function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \quad x \ \ rational \\ 0, & 0 \leq x \leq 1, \quad x \ \ irrational \end{cases}$$

is <u>not</u> Riemann integrable on [0,1].

Solution:

Let $P = \{[x_0,x_1],[x_1,x_2],\dots,[x_{i-1},x_i],\dots,[x_{n-1},x_n]\}$ be any partition of [0,1]. Then

$$L(f,P) = \sum_{i=1}^{n} m_i \delta x_i = \sum_{i=1}^{n} 0 \times \delta x_i = 0 \qquad \Rightarrow \quad \underline{\int_0^1} f = 0$$

and

$$U(f,P)=\sum_{i=1}^{n} M_i \delta x_i = \sum_{i=1}^{n} 1 \times \delta x_i = \sum_{i=1}^{n} \delta x_i = 1 \qquad \Rightarrow \quad \overline{\int_0^1} f = 1$$

Since $\underline{\int_0^1} f \neq \overline{\int_0^1} f$

$\Rightarrow$ f is not Riemann integrable on $[0,1]$ (or $f \notin \mathcal{R}[0,1]$).

Example (2.6): Let $f:[-1,1] \to R$ defined as

$$f(x)=\begin{cases} 2 & if \quad x<0 \\ 3 & if \quad x \geq 0 \end{cases}$$

Prove that f is Riemann integrable on $[-1,1]$.

Solution:

Let $P=\left\{\left[-1,-\frac{1}{n}\right],\left[-\frac{1}{n},\frac{1}{n}\right],\left[\frac{1}{n},1\right]\right\}$

We have

$\delta x_1 = -\frac{1}{n}-(-1) = 1-\frac{1}{n}$, $\delta x_2 = \frac{1}{n}-(-\frac{1}{n}) = \frac{2}{n}$, $\delta x_3 = 1-\frac{1}{n}$.

and

$m_1 = inf\left\{f(x): x \in \left[-1,-\frac{1}{n}\right]\right\} = 2$, $M_1 = sup\left\{f(x): x \in \left[-1,-\frac{1}{n}\right]\right\} = 2$

$m_2 = inf\left\{f(x): x \in \left[-\frac{1}{n},\frac{1}{n}\right]\right\} = 2$, $M_2 = sup\left\{f(x): x \in \left[-\frac{1}{n},\frac{1}{n}\right]\right\} = 3$

$m_3 = inf\left\{f(x): x \in \left[\frac{1}{n},1\right]\right\} = 3$, $M_3 = sup\left\{f(x): x \in \left[\frac{1}{n},1\right]\right\} = 3$

It follows that

$$\begin{aligned} L(f,P) = \textstyle\sum_{i=1}^{3} m_i \delta x_i &= m_1\delta x_1 + m_2 \delta x_2 + m_3 \delta x_3 \\ &= 2 \times (1-\tfrac{1}{n}) + 2 \times \tfrac{2}{n} + 3 \times (1-\tfrac{1}{n}) \\ &= 2-\tfrac{2}{n}+\tfrac{4}{n}+3-\tfrac{3}{n} = 5-\tfrac{1}{n} \end{aligned}$$

$U(f,P) = \sum_{i=1}^{3} M_i \delta x_i = M_1\delta x_1 + M_2 \delta x_2 + M_3 \delta x_3$

$$= 2 \times (1 - \tfrac{1}{n}) + 3 \times \tfrac{2}{n} + 3 \times (1 - \tfrac{1}{n})$$

$$= 2 - \tfrac{2}{n} + \tfrac{6}{n} + 3 - \tfrac{3}{n} = 5 + \tfrac{1}{n}$$

We get

$$\underline{\int_{-1}^{1}} f = \sup_P L(f, P) = \sup\{4, 4.5, 4.666\overline{6}\ , 4.75\ , 4.8, 4.833\overline{3}, \dots, 5\} = 5$$

$$\overline{\int_{-1}^{1}} f = \inf_P U(f, P) = \inf\{6, 5.5, 5.333\overline{3}, 5.25, 5.2, 5.166\overline{6}, \dots, 5\} = 5$$

$$\Rightarrow \quad \int_{-1}^{1} f = \underline{\int_{-1}^{1}} f = \overline{\int_{-1}^{1}} f = 5$$

$\therefore$ f is Riemann integrable on $[-1,1]$, (or $f \in \mathcal{R}[-1,1]$).

Example (2.7): Let $f: [-2,2] \to R$ defined as

$$f(x) = \begin{cases} 1 & if \quad -2 \le x \le -1 \\ 3 & if \quad -1 < x \le 1 \\ 5 & if \quad 1 < x \le 2 \end{cases}$$

Prove that f is Riemann integrable on $[-2,2]$.

Solution: Let

$P = \left\{\left[-2, -1 - \frac{1}{n}\right], \left[-1 - \frac{1}{n}, -1 + \frac{1}{n}\right], \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right], \left[1 + \frac{1}{n}, 2\right]\right\}$

$\delta x_1 = -1 - \frac{1}{n} - (-2) = 1 - \frac{1}{n}$, $\delta x_2 = -1 + \frac{1}{n} - (-1 - \frac{1}{n}) = \frac{2}{n}$,

$\delta x_3 = 1 - \frac{1}{n} - (-1 + \frac{1}{n}) = 2 - \frac{2}{n}$, $\delta x_4 = 1 + \frac{1}{n} - (1 - \frac{1}{n}) = \frac{2}{n}$,

$\delta x_5 = 2 - (1 + \frac{1}{n}) = 1 - \frac{1}{n}$

We have

$m_1 = \inf\left\{f(x): x \in \left[-2, -1 - \frac{1}{n}\right]\right\} = 1,$

$m_2 = inf\left\{f(x): x \in \left[-1-\frac{1}{n}, -1+\frac{1}{n}\right]\right\} = 1,$

$m_3 = inf\left\{f(x): x \in \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]\right\} = 3,$

$m_4 = inf\left\{f(x): x \in \left[1-\frac{1}{n}, 1+\frac{1}{n}\right]\right\} = 3,$

$m_5 = inf\left\{f(x): x \in \left[1+\frac{1}{n}, 2\right]\right\} = 5,$

and

$M_1 = sup\left\{f(x): x \in \left[-2, -1-\frac{1}{n}\right]\right\} = 1$

$M_2 = sup\left\{f(x): x \in \left[-1-\frac{1}{n}, -1+\frac{1}{n}\right]\right\} = 3$

$M_3 = sup\left\{f(x): x \in \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]\right\} = 3$

$M_4 = sup\left\{f(x): x \in \left[1-\frac{1}{n}, 1+\frac{1}{n}\right]\right\} = 5$

$M_5 = sup\left\{f(x): x \in \left[1+\frac{1}{n}, 2\right]\right\} = 5$

It follows that

$$L(f,P) = \sum_{i=1}^{5} m_i \delta x_i$$
$$= m_1\delta x_1 + m_2\delta x_2 + m_3\delta x_3 + m_4\delta x_4 + m_5\delta x_5$$
$$= 1 \times (1-\frac{1}{n}) + 1 \times \frac{2}{n} + 3 \times (2-\frac{2}{n}) + 3 \times \frac{2}{n} + 5 \times (1-\frac{1}{n})$$
$$= 1 - \frac{1}{n} + \frac{2}{n} + 6 - \frac{6}{n} + \frac{6}{n} + 5 - \frac{5}{n} = 12 - \frac{4}{n}$$

$$U(f,P) = \sum_{i=1}^{5} M_i \delta x_i$$
$$= M_1\delta x_1 + M_2\delta x_2 + M_3\delta x_3 + M_4\delta x_4 + M_5\delta x_5$$
$$= 1 \times (1-\frac{1}{n}) + 3 \times \frac{2}{n} + 3 \times (2-\frac{2}{n}) + 5 \times \frac{2}{n} + 5 \times (1-\frac{1}{n})$$
$$= 1 - \frac{1}{n} + \frac{6}{n} + 6 - \frac{6}{n} + \frac{10}{n} + 5 - \frac{5}{n} = 12 + \frac{4}{n}$$

We get