Def.:- If R is a relation from set A into A the identity relation is defined by

$$I_A = \{(x,y) : (x,y) \in A \times A \text{ and } x = y\}$$

Ex.:- let $A = \{a,b\}$

$$R = \{(a,a),(a,b),(b,a),(b,b)\}$$

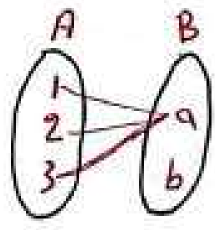
$$\therefore I_A = \{(a,a),(b,b)\}.$$ نعين العناصر المتشابهة

Def.:- let R be relation from A into B, then

$$\text{domain } R = \{x : \exists y \in B \ni (x,y) \in R\} = D_R$$

$$\text{Range } R = \{y : \exists x \in A \ni (x,y) \in R\} = R_R$$

Ex.:- let $A = \{1,2,3\}$, $B = \{a,b\}$



$$R = \{(1,a),(2,a),(3,a)\}$$

$$D_R = \{1,2,3\} \quad , \quad R_R = \{a\}$$

Th. If R is a relation from A into B, then

① $\text{Domain } R = \text{Rang}(R^{-1})$

② $\text{Rang } R = \text{Domain}(R^{-1})$

$\Rightarrow$

[illegible]

ملاحظة: R.D [illegible]

Proof (1): Let $x \in D_R \Leftrightarrow \exists y$ s.t $(x,y) \in R$

$\Leftrightarrow \exists y$ s.t $(y,x) \in R^{-1} \Leftrightarrow x \in R_{R^{-1}}$

$\Leftrightarrow D_R \subseteq R_{R^{-1}}$ and $R_{R^{-1}} \subseteq D_R$

$\therefore D_R = R_{R^{-1}}$

Proof (2): Let $y \in R_R \Leftrightarrow \exists x$ s.t $(x,y) \in R$

$\Leftrightarrow \exists x$ s.t $(y,x) \in R^{-1} \Leftrightarrow y \in D_{R^{-1}}$

$\Rightarrow R_R \subseteq D_{R^{-1}}$ and $D_{R^{-1}} \subseteq R_R$

$\therefore R_R = D_{R^{-1}}$

Def.: Let R be a relation A, انواع العلاقات

then ① R is called (reflexive) انعكاسية if $\forall x \in A$.

$(x,x) \in R$ iff $I_A \subseteq R$.

أبسط مثال على العلاقة الانعكاسية هي المساواة [illegible]

② R is called (symmetric) متناظرة if $(x,y) \in R$, then $(y,x) \in R$

مثال عليها علاقة التوازي [illegible]

③ R is called (transitive) متعدية if $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$.

مثال على علاقة التعدي هي علاقة الأكبر

$3 > 2 > 1$

$\Rightarrow 3 > 1$

Def.:- A relation R is called (equivalence) if R is reflexive, symmetric, transitive.

Ex.:- $R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x = y\}$

Sol.:- $R = \{\ldots, (-4,-4), (-3,-3), (-2,-2), (-1,-1), (0,0), (1,1), (2,2), \ldots\}$

① R is reflexive since $\forall x \in \mathbb{Z}$, $(x,x) \in R$.

② R is symmetric since $(x,y) \in R \Rightarrow (y,x) \in R$.

③ R is transitive $(\overset{x}{-1},\overset{y}{-1})(\overset{y}{-1},\overset{z}{-1}) \Rightarrow (-1,-1) \in R$

ملاحظة:- اذا كانت العلاقة مساواة فقط فانها تكون علاقة تكافؤ

$\therefore R$ is equivalence

$D_R = \mathbb{Z}$, $R_R = \mathbb{Z}$.

Ex.:- $R = \{(a,b) \in \mathbb{N} \times \mathbb{N} ; a \perp b\}$

كل مستقيم عمودي على نفسه

Sol.:- ① $a \perp a \Rightarrow R$ is reflexive.

② $a \perp b \Rightarrow b \perp a$

$(a,b) \in R \Rightarrow (b,a) \in R \Rightarrow R$ is symmetric.

③ $a \perp b$ and $b \perp c \not\Rightarrow a \perp c \Rightarrow R$ is not transitive

$(a,b) \in R$ $(b,c) \in R$ $(a,c) \notin R$

$\therefore R$ is not equivalence.

Ex.:- $R = \{(a,b) \in N \times N : a \parallel b\}$

Sol.:- ① $b_1 \parallel b_1$ reflexive.

② $L_1 \parallel L_2 \Rightarrow L_2 \parallel L_1$ symmetric.

③ $L_1 \parallel L_2$ and $L_2 \parallel L_3 \Rightarrow L_1 \parallel L_3$ transitive.

Ex.:- $R = \{(x,y) \in Z \times Z : x \leq y\}$

$R = \underset{(0,0),(1,1),(2,2),(-1,-1),\dots}{I_Z} \cup \{\dots, (-4,-3), \dots, (-2,-1), \dots (0,1), \dots (1,2), \dots, (2,3), (1,3), \dots\}$

① R is reflexive since $I_Z \subseteq R$.

② R is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$.

③ R is transitive since $(1,2) \in R$ and $(2,3) \in R$
$\Rightarrow (1,3) \in R$.

$\therefore$ R is not equivalence.

Example:- Prove your answer in each case, equivalence relation or neither.

① $R = \{(x,y) \in Z \times Z : \frac{x-y}{5}\}$

② $R = \{(x,y) \in Z \times Z : x < y\}$ H.w

$R = \{(1,2), (1,3), \dots, (2,3), \dots, (3,4), \dots, (-1,2), (-2,-1), (0,1), (3,4), \dots\}$

③ $R = \{(x,y) \in Z \times Z : x > y\}$ H.w

$R = \{\dots, (-3,-4), \dots, (-1,-2), \dots, (1,0), \dots (2,1), \dots (3,2), (3,1), \dots\}$

Sol.:- ① let $x=y \Rightarrow \frac{x-x}{5} = \frac{0}{5} = 0 \in Z$

$\Rightarrow (x,x) \in Z \times Z \ , \forall x \in Z$ $\therefore R$ is ref.

② let $(x,y) \in R \Rightarrow \frac{x-y}{5} \in Z \Rightarrow \frac{-y+x}{5} = \frac{-(y-x)}{5} \in Z$

$\Rightarrow (y,x) \in R$ $\therefore R$ is sym.

③ let $(x,y) \in R$ and $(y,z) \in R$

$\Rightarrow \frac{x-y}{5} + \frac{y-z}{5} = \frac{x-z}{5} \in Z$

$\Rightarrow (x,z) \in R$ $\therefore R$ is tra.

$\therefore R$ is eq.

Ex.:- $R = \{ I_A \cup (a,b), (b,a) \}$

I_A: $(a,a), (b,b)$

Sol.:① R is ref. $\forall x \in A$, $(x,x) \in R$

② R is sym. if $(a,b) \in R \Rightarrow (b,a) \in R$

③ R is tra. $(a,b) \in R$, $(b,a) \in R$

$(a,a) \in R$

Ex.:- $R = \{(1,2),(2,1),(7,9)\} \cup I_Z$

Sol.:-① R is reflexive, since $(x,x) \in R$, $\forall x \in Z$

② R is not symmetric, since $(7,9) \in R$ but $(9,7) \notin R$

③ R is transitive, since $(1,2) \in R$ and $(2,1) \in R$

$\Rightarrow (1,1) \in R$

R is not equivalence relation.