3- let $(a,b),(c,d),(e,f) \in Z \times Z$ $\quad a,b,c,d,e,f \in Z$

s.t. $\quad (a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$

$\times e \; (ad = bc)$ and $(cf = de) \times a$

$ead = ebc$ and $acf = ade$

$\cancel{e}ad = ebc \qquad acf = \cancel{e}ad$

$\Rightarrow ebc = acf \Rightarrow eb\cancel{c} = af\cancel{c}$

$\Rightarrow af = be$

$\Rightarrow (a,b)\,R\,(e,f) \quad \therefore R$ is transitive.

$\therefore R$ is equivalence relation on $Z \times Z$.

Definition:- The set of all equivalence classes of R in $Z \times Z$ is said to be Rational Numbers.

i.e. $Q = \{ [n_1, n_2] : n_1, n_2 \in Z \,,\, n_2 \neq 0 \}$ بشرط

Ex.:- $0 = [0, n] \;,\; n \in Z \,,\, n \neq 0$ $\quad \left(0 = \frac{0}{n}\right)$

$\left(\frac{n}{n}\right)$ $\quad 1 = [n, n] \;,\; n \in Z \,,\, n \neq 0$

Definition:- If $a, b \in Q$ and $a = [a_1, a_2]\,,\, b = [b_1, b_2]$

$\forall a_1, a_2, b_1, b_2 \in Z \,,\, a_2 \neq 0 \,,\, b_2 \neq 0$. Then

1. $a + b = [a_1, a_2] + [b_1, b_2] = \left(\frac{a_1}{a_2} + \frac{b_1}{b_2} \right) = \frac{a_1 b_2 + b_1 a_2}{a_2 b_2}$

$= [a_1 b_2 + b_1 a_2 \,,\, a_2 b_2]$

2. $ab = [a_1, a_2][b_1, b_2]$

$= \frac{a_1}{a_2} \frac{b_1}{b_2} = \frac{a_1 b_1}{a_2 b_2} = [a_1 b_1 \,,\, a_2 b_2]$.

Note :- $[a,b] = [ca,cb] = [ea,eb]$, $\forall a,b,c,d,e,f \in \mathbb{Z}$ (23)

Ex. :- $[3,6] = [9,18] = [12,24] = [15,30] = [1,2] = \frac{1}{2}$.

Properties of Rational number :-

عملية الجمع بالاعداد النسبية تمتاز بالعملية التجميعية

1. $(a+b)+c = a+(b+c) \quad \forall a,b,c \in \mathbb{Q}$
2. $a+b = b+a \quad \forall a,b \in \mathbb{Q}$.
3. $\exists\, 0 \in \mathbb{Q}$ s.t. $0+a = a+0 = a$, $\forall a \in \mathbb{Q}$ العنصر المحايد الجمعي
4. $\forall a \in \mathbb{Q}$, $\exists -a \in \mathbb{Q}$ s.t. $a+(-a) = -a+a = 0$.
   يعني العنصر تجمعه مع معكوسه الناتج صفر
5. $(ab)c = a(bc)$.
6. $ab = ba$.
7. $\exists\, 1 \in \mathbb{Q}$ s.t. $1.a = a.1 = a$, $\forall a \in \mathbb{Q}$.
8. $\forall a \in \mathbb{Q}$, $\exists\, a^{-1} \in \mathbb{Q}$ s.t. $aa^{-1} = a^{-1}a = 1$.
   معكوس العدد
9. $a(b+c) = ab+ac$.

وهذه الا (a,b) اعداد نسبية مركبتها اعداد صحيحة

Proof(2) :- let $a = [a_1, a_2]$, $b = [b_1, b_2]$, $a_1, a_2, b_1, b_2 \in \mathbb{Z}$, $a_2 \neq 0$, $b_2 \neq 0$

$$a+b = [a_1, a_2] + [b_1, b_2]$$

$$= \left(\frac{a_1}{a_2} + \frac{b_1}{b_2}\right) = \left(\frac{a_1 b_2 + b_1 a_2}{a_2 b_2}\right)$$

$$= \left(\frac{b_2 a_1 + a_2 b_1}{a_2 b_2}\right) = \left(\frac{b_1}{b_2} + \frac{a_1}{a_2}\right)$$

$$= [b_1, b_2] + [a_1, a_2] = b + a.$$

Proof(3):- we claim that $0 = [0, m], m \in \mathbb{Z}, m \neq 0$

let $a = [a_1, a_2]$, $a_1, a_2 \in \mathbb{Z}$, $a_2 \neq 0$

$$a + 0 = [a_1, a_2] + [0, m] = \left(\frac{a_1}{a_2} + \frac{0}{m}\right) = \left(\frac{a_1 m + 0 a_2}{a_2 m}\right)$$

$$= \left(\frac{a_1 m}{a_2 m}\right) = \left(\frac{a_1}{a_2}\right) = [a_1, a_2] = a$$

to prove $0 = [0, m]$ is unique

let $0 = [0, m]$, $0 = [0, n]$, $n \in \mathbb{Z}$, $n \neq 0$.

$\because 0n = m0 \Rightarrow (0, n) R (0, m) \Rightarrow [0, n] = [0, m]$.

Proof(4):- let $a = [a_1, a_2]$, $a_1, a_2 \in \mathbb{Z}$, $a_2 \neq 0$, we claim that $-a = [-a_1, a_2]$

then $a + (-a) = [a_1, a_2] + [-a_1, a_2] = \left(\frac{a_1}{a_2}\right) + \left(\frac{-a_1}{a_2}\right)$

مركبة أولى

$$= \left(\frac{a_1 a_2 - a_2 a_1}{a_2^2}\right) = [a_1 a_2 - a_2 a_1, a_2^2]$$

مركبة ثانية

$$= [0, a_2^2] = 0.$$

Proof(7):- we claim that $1 = [m, m]$, $m \in \mathbb{Z}$, $m \neq 0$

(كيف يتم الحصول على العدد واحد بالأعداد النسبية من خلال قسمة العدد على نفسه)

let $a = [a_1, a_2]$, $a_1, a_2 \in \mathbb{Z}$, $a_2 \neq 0$

$$a \cdot 1 = [a_1, a_2][m, m] = \left(\frac{a_1}{a_2}\right)\left(\frac{m}{m}\right)$$

$$= \left(\frac{a_1 m}{a_2 m}\right) = \left(\frac{a_1}{a_2}\right) = [a_1, a_2] = a.$$

Now, to prove $1 = [m, m]$ is a unique.

let $1 = [n, n]$, $n \in \mathbb{Z}$, $n \neq 0$

$\because nm = mn$

$\Rightarrow (n, n) R (m, m)$

$\Rightarrow [n, n] = [m, m] \Rightarrow 1$ is unique.

Def.:- let $x, y \in \mathbb{Q}$ and $x = [a_1, a_2]$, $y = [b_1, b_2]$

$a_1, a_2, b_1, b_2 \in \mathbb{Z}$, $a_2, b_2 \neq 0$, then

$x - y = x + (-y) = [a_1, a_2] + [-b_1, b_2]$ $\quad \frac{a_1}{a_2} + \frac{-b_1}{b_2}$

عملية الجمع بالاعتماد
النسبة
$= [a_1 b_2 - b_1 a_2, a_2 b_2]$

Ex.:- let $x = [2, 4]$, $y = [3, 9]$

اصل هذا العدد $\frac{1}{2}$

$\Rightarrow x - y = [2, 4] + [-3, 9]$

$= [18 - 12, 36] = [6, 36] = \frac{6}{63} = \frac{1}{6}$.

$x - y = [2, 4] + [-3, 9] = (\frac{2}{4} + \frac{-3}{9}) = \frac{2(9) + 4(-3)}{36} = \frac{18 - 12}{36}$

$= [18 - 12, 36] = [6, 36] = \frac{6}{36} = \frac{1}{6}$

Definition:- let $x \in \mathbb{Q}$ and $x \neq 0$ with $x = [m, n]$

$m, n \in \mathbb{Z}$ and $n \neq 0$, then المركبة الاولى × المركبة الثانية

x is said to be positive iff $mn > 0$

and

x is said to be negative iff $mn < 0$.

Example:- If $x, y \in Q^+$, then $x+y \in Q^+$.

Proof:- let $x=[a,b]$ and $y=[c,d]$, $a,b,c,d \in Z$, $b,d \neq 0$

$\therefore x+y=[a,b]+[c,d]=[ad+bc, bd]$

مطلوب ضرب المركبة الأولى × المركبة الثانية

we must prove that $(ad+bc)bd>0$

$\because x \in Q^+ \Rightarrow ab>0 \xRightarrow{*d^2} abd^2>0$

$\because y \in Q^+ \Rightarrow cd>0 \xRightarrow{*b^2} cdb^2>0$

العامل b إذا اعداد موجبة او سالبة بالتربيع يبقى موجب

$\therefore x+y = abd^2 + cdb^2 > 0$

$= (ad+cb)bd>0$

$\therefore x+y>0 \Rightarrow x+y \in Q^+$.

Example:- If $x, y \in Q^+$, then $xy \in Q^+$.

Proof:- let $x=[a,b]$, $y=[c,d]$, $a,b,c,d \in Z$, $b,d \neq 0$.

$xy=[a,b][c,d]=[ac,bd]$

to prove that $(ac)bd>0$

$\because x \in Q^+ \Rightarrow ab>0$

$\because y \in Q^+ \Rightarrow cd>0$

$\therefore xy = (ab)cd>0$

$= (ac)bd>0$

عملية الضرب بالاعداد الصحيحة تبديلية

$\therefore xy>0 \Rightarrow xy \in Q^+$.