Example (4.14):

(1) $A = \{1,2,3\}$

$\forall\ x \in A,\ B_\varepsilon(x) \ni x\ ; (B_\varepsilon(x) \cap A)\{x\} = \emptyset \quad \Rightarrow \quad A' = \emptyset$

$\Rightarrow\ \bar{A} = A \cup A' = \{1,2,3\}$

(2) $A = Q$

$\Rightarrow\ Q' = R$

$\Rightarrow\ \bar{Q} = Q \cup Q' = Q \cup R = R$

Definition (4.8): (Exterior Set) المجموعة الخارجية

Let (X,d) be a metric space and $A \subset X$, the **exterior set** of A, is defined as: $A^e = A^{c^\circ}$.

Example (4.15): Let (R,d) be a metric space and $A = (1,5]$

$A^c = (-\infty, 1] \cup (5, \infty)$

$A^e = A^{c^\circ} = (-\infty, 1) \cup (5, \infty)$

Definition (4.9): (Boundary Set) مجموعة الحدود

Let (X,d) be a metric space and $A \subset X$, the **boundary set** of A, is defined as: $A^b = \bar{A} - A^\circ$.

Example (4.16): Let (R,d) be a metric space and $A = (0,2)$

$A^\circ = (0,2)$

$\bar{A} = [0,2]$

$A^b = \bar{A} - A^\circ = [0,2] - (0,2) = \{0,2\}$

Definition (4.10): Let (X,d) be a metric space. Let $< x_n >$ be a sequence in X. We say that $< x_n >$ is **convergent** to x_0 if

$\forall\ \varepsilon > 0\ , \exists\ k \in N \quad s.t.\ \ d(x_n, x_0) < \varepsilon\ ,\ \forall\ n > k\ .$

i.e. $\lim_{n\to\infty} x_n = x_0$ or $x_n \to x_0$

If no such number x_0 exists, we say that $< x_n >$ diverges.

Proposition (4.1): If $< x_n >$ is convergent, and $x_n \to x_0$ then x_0 is unique.

Proof: Assume $x_0 \neq y_0$ and $x_n \to x_0$, $x_n \to y_0$

Assume $d(x_0, y_0) = r$

Let $B_{\frac{r}{2}}(x_0)$ and $B_{\frac{r}{2}}(y_0)$

Now, $B_{\frac{r}{2}}(x_0) \cap B_{\frac{r}{2}}(y_0) = \emptyset$

By definition of convergent $B_{\frac{r}{2}}(x_0)$ contains almost element of $< x_n >$

And $B_{\frac{r}{2}}(y_0)$ contains almost element of $< x_n >$ C!

Thus, $x_0 = y_0$

$\Rightarrow$ The convergent point is unique

Definition (4.11): Let (X, d) be a metric space. Then the sequence $< x_n >$ is said to be **bounded** iff $\exists\, M \in R$ s.t. $d(x_n, 0) \leq M, \forall\, n$.

Definition (4.12): Let (X, d) be a metric space. The sequence $< x_n >$ is called a **Cauchy sequence (essential sequence)** if for every $\forall\, \varepsilon > 0$, $\exists\ k \in N$ s.t. $d(x_m, x_n) < \varepsilon, \forall\, m, n > k$.

Proposition (4.2): If (X, d) is a metric space, then every convergent sequence is Cauchy sequence.

Proof:

Let $< x_n >$ be a convergent sequence i.e. $x_n \to x_0$.

Let $\varepsilon > 0$ then $\exists\, k \in N$ s.t. $d(x_n, x_0) < \frac{\varepsilon}{2}, \forall\, n > k$

Now, $d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m)$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow\ d(x_n, x_m) < \varepsilon\ , \forall\ n, m > k$

$\therefore\ < x_n >$ is Cauchy sequence

Note (4.4): The converse of the above proposition is not true as shown in the following example.

Example (4.17):

(1) Let $X = R/\{0\}$, d is the absolute value the $< \frac{1}{n} >$ is Cauchy sequence but not convergent in X.

(2) Let $X = Q$, and $< x_n > = \{1,1.4,1.41,1.414, \ldots\} \rightarrow \sqrt{2} \notin Q$ then $< x_n >$ is Cauchy sequence but not convergent in X.

Definition (4.13): A space is **complete** if every Cauchy sequence is convergent.

Theorem (4.4): The Euclidian spaces R^k are complete $\forall\ k$.

Proof:

It is enough to prove that it is true for $k = 2$

Let $< z_m >$ be Cauchy sequence in R^2

Let $\varepsilon > 0$, assume $z_m = (x_m, y_m)$, where $x_m, y_m \in R$

Since $< z_m >$ is Cauchy sequence

$\exists\ k \in N$ s.t. $d(z_m, z_n) < \varepsilon\ , \forall\ m, n > k$

i.e. $\sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} < \varepsilon$

$\Rightarrow\ (x_m - x_n)^2 + (y_m - y_n)^2 < \varepsilon^2$

$\Rightarrow\ |x_m - x_n| < \varepsilon$ and $|y_m - y_n| < \varepsilon$, $\forall\ m, n > k$

Since $< x_m >$ and $< y_m >$ are Cauchy Sequence in R and

Since R is a complete space

$\Rightarrow\ < x_m >$ and $< y_m >$ are convergent

Therefore, $\exists\ x_0, y_0 \in R$ s.t. $x_m \rightarrow x_0$ and $y_m \rightarrow y_0$

i.e. $\exists\, k \in N$ s.t. $|x_m - x_0| < \frac{\varepsilon}{2}$ and $|y_m - y_0| < \frac{\varepsilon}{2}$, $\forall\, m > k$

Let $z_0 = (x_0, y_0)$

$\Rightarrow\ \left(d(z_m, z_0)\right)^2 = (x_m - x_0)^2 + (y_m - y_0)^2$

$< \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{2}$

$\Rightarrow\ d(z_m, z_0) < \varepsilon$, $\forall\, m > k$

$\Rightarrow\ < z_m >$ is a convergent

$\therefore\ R^2$ is complete metric space.

Definition (4.14): (Contraction Mapping) التطبيق الانكماشي

Let (X, d) be a metric space. A mapping $f: X \to X$ is a **contraction mapping** if $\exists$ constant c, with $0 \le c < 1$, s.t. $d\big(f(x), f(y)\big) \le c d(x, y)$, $\forall\, x, y \in X$.

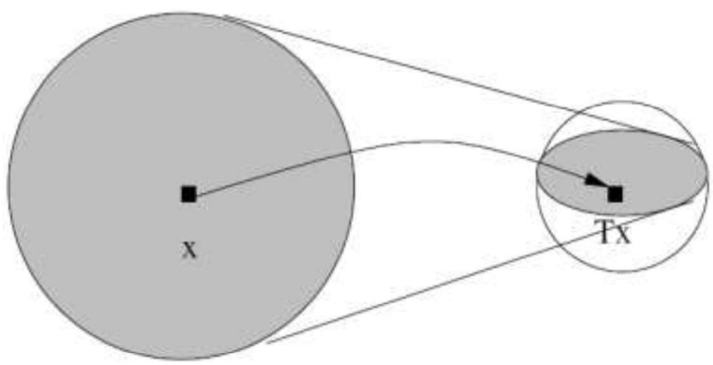

Figure (4.1). f is a contraction mapping

Theorem (4.5): (Contraction Mapping Theorem) نظرية التطبيق الانكماشي

Let (X, d) be a complete metric space. Let $f: X \to X$ is a contraction mapping, then $\exists$ unique x such that $f(x) = x$.

Proof:

Let x_0 be any point in X, we define a sequence $< x_n >$ in X by

$x_n = f(x_{n-1})$, for each $n \in N$

i.e. $x_1 = f(x_0)$

$x_2 = f(x_1) = f\big(f(x_0)\big) = f^2(x_0)$